Periodic bifurcation points of nonlinear nonautonomous measure differential equations

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This contribution is devoted to periodic problems for differential like equations of the form

$$Dx = f(\lambda, x, t) + g(x, t) \,.\, Du, \quad x(0) = x(T),$$

$$x(t) = x(T) + \int_0^t f(\lambda, x(s), s)\, ds + \int_0^t g(x(s), s)\, du(s),$$

$$x(t) = x(T) + \int_0^t DF(x(\tau), \sigma) \quad \text{or} \quad \frac{dx}{d\tau} = DF(x, t)$$

Content

- $-\infty < a < b < \infty$,
- $f\colon [a,b] \to \mathbb{R}^n$ is **regulated** on $[a,b]$ if
  $f(s+) := \lim_{\tau \to s+} f(\tau) \in X$ for $s \in [a,b)$, $f(t-) := \lim_{\tau \to t-} f(\tau) \in X$ for $t \in (a,b]$,
- $\Delta^+ f(s) = f(s+) - f(s)$, $\Delta^- f(t) = f(t) - f(t-)$, $\Delta f(t) = f(t+) - f(t-)$.
- $\mathrm{G}[a,b] = \{f\colon [a,b] \to \mathbb{R}^n;\ f$ is regulated on $[a,b]\}$.
  ($\mathrm{G}[a,b]$ is Banach space with respect to the norm $\|f\|_\infty = \sup_{t \in [a,b]} \|f(t)\|$).
  - regulated functions are uniform limits of finite step functions,
  - regulated functions have at most countably many points of discontinuity.
- $\mathrm{BV}[a,b] = \{f\colon [a,b] \to \mathbb{R}^n\colon \mathrm{var}_a^b f < \infty\}$
  is the space of functions with **bounded variation** on $[a,b]$.

  $BV[a,b] \subset G[a,b]$.

We will consider periodic problem for distributional (measure) differential system

$$Dx = f(\lambda, x, t) + g(x, t)\,.\,Du, \quad x(0) = x(T). \tag{P}$$

Basic assumptions (A)

- $T > 0$, $\Omega \subset \mathbb{R}^n$ and $\Lambda \subset \mathbb{R}$ are open,
- $f : \Lambda \times \Omega \times [0, T] \to \mathbb{R}^n$, $g : \Omega \times [0, T] \to \mathbb{R}^n$,
- $u : \mathbb{R} \to \mathbb{R}$ is BV on $[0, T]$ and left-continuous, not necessarily monotonous
- $u(0-) = u(0), \quad u(T+) = u(T)$,
- $x : [0, T] \to \mathbb{R}^n$,
- Dx and Du are distributional derivatives of x and u, respectively,
- $g(x, t)\,.\,Du$ stands for the distributional product.

Test functions: $\mathcal{D} = \{\varphi : \mathbb{R} \to \mathbb{R}^n : \varphi \in C^\infty(\mathbb{R}), \varphi^{(j)}(t) = 0 \text{ for } t \notin (0, T)\}$.

$$\varphi_k \to \varphi_0 \iff \lim_k \|\varphi_k^{(j)} - \varphi_0^{(j)}\| = 0 \text{ for } j \in N \cup \{0\}.$$

Distributions: linear continuous functionals on $\mathcal{D}$.

- $\mathcal{D}^*$ is the space of distributions on $[0, T]$.
- $f \in \mathcal{D}^*, \varphi \in \mathcal{D} \Rightarrow \; < f, \varphi >$ is the value of f on φ.
- $f \in L_1[0, T] \quad \Rightarrow \quad < f, \varphi > := \varphi \in \mathcal{D} \to \int_0^T f(t)\,\varphi(t)\,dt \Rightarrow f \in \mathcal{D}^*$.
- $0 \in \mathcal{D}^*$ is a measurable function vanishing a.e. on $[0, T]$.
- $f \in G[0, T]$ left-continuous on $(0, T]$ $\Rightarrow$ $f = 0 \in \mathcal{D}^*$ iff $f(t) \equiv 0$.
- For $f \in \mathcal{D}^*$, the symbol Df stands for its distributional derivative, i.e.

  $$Df : \varphi \in \mathcal{D} \to < Df, \varphi > = - < f, \varphi' > \quad \text{for } \varphi \in \mathcal{D}.$$
- $f \in AC[0, T] \Rightarrow Df = f'$.
- $Df = 0$ iff $f \in L_1^n[0, T]$ and $\exists\, c \in \mathbb{R}^n$ such that $f(t) = c$ a.e. on $[0, T]$.

$$Dx = f(\lambda, x, t) + g(x, t)\,.\,Du, \quad x(0) = x(T), \tag{P}$$

Definition

$x : [0, T] \to \mathbb{R}^n$ is a solution of (P) if:

- $x \in G[0, T]$ is left-continuous on $(0, T]$, $x(t) \in \Omega$ for $t \in [0, T]$,
- distributional product $(g \circ x)\,.\,Du$ has a sense, $x(0) = x(T)$ and

$$< Dx, \varphi > = < (f \circ x), \varphi > + < (g \circ x)\,.\,Du, \varphi > \quad \text{for all } \varphi \in \mathcal{D}.$$

$$x : [0, T] \to \mathbb{R}^n \quad \Rightarrow \quad (f \circ x)(t) = f(\lambda, x(t), t), \quad (g \circ x) = g(x(t), t)$$

Distributional product

- $f \in \mathcal{D}^*,\ g \in \mathcal{D} \Rightarrow \langle f.g, \varphi \rangle = \langle f, g\,\varphi \rangle$ for $\varphi \in \mathcal{D}$.
- $f\,g \in L^1\,[0, T] \Rightarrow \langle f.g, \varphi \rangle = \int_0^T f(t)\,g(t)\,\varphi(t)\,dt$ for $\varphi \in \mathcal{D}^n$.

In the space of distributions we do not have any general definition of a product

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Definition

If the KS integral $\displaystyle\int_0^T g\,d\,h \in \mathbb{R}^n$ exists, we define $g\,.\,Dh = DH$, where $\displaystyle H(t) := \int_0^t g\,d\,h$.

- $g\,.\,Dh$ is well-defined if $g, h \in G\,[0,T]$ and at least one of them is in BV.
- $D(f.g) = Df.g + f\,.\,Dg + Df.\Delta^+\widetilde{g} - \Delta^-\widetilde{f}\,.\,Dg$, where

$$\Delta^+\widetilde{g}(t) = \begin{cases} \Delta^+ g(t) & \text{if } t < T, \\ 0 & \text{if } t = T \end{cases} \quad \text{and} \quad \Delta^-\widetilde{f}(t) = \begin{cases} 0 & \text{if } t = 0, \\ \Delta^- f(t) & \text{if } t > 0. \end{cases}$$

Integral version

$$Dx = f(\lambda, x, t) + g(x, t)\,.\,Du, \quad x(0) = x(T), \tag{P}$$

$$x(t) = x(T) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,du(s) \tag{I}$$

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$$x(t) = x(T) + (L)\int_0^t f(\lambda, x(s), s)\, ds + (LS)\int_{[0,t)} g(x(s), s)\, d\mu_u, \tag{L}$$

(μ_u is a signed measure generated by $u \in BV\,[0, T]$)

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[*G.A. Monteiro, A. Slavík & M. T.: Kurzweil-Stieltjes integral.*, Sec.6.12] $\Longrightarrow$

$$(LS)\int_{[0,T)} g\, d\mu_u \text{ exists } \Rightarrow \int_0^T g\, du = (LS)\int_{[0,T)} g\, d\mu_u,$$

$\Longrightarrow$ (L) is a special case of (I).

$$x(t) = x(T) + \int_0^t f(\lambda, x(s), s)\, ds + \int_0^t g(x(s), s)\, du(s) \qquad \text{(I)}$$

Proposition

Assume: (A) and

$$\text{(B)} \quad \begin{cases} f(\lambda, \cdot, \cdot) \text{ is Carathéodory on } \Omega \times [0, T] \text{ for any } \lambda \in \Lambda; \\ g(\cdot, t) \text{ is continuous on } \Omega \text{ for } t \in [0, T] \text{ and there is } m_u \text{ such that:} \\ \int_0^T m_u(s)\, d[\operatorname{var}_0^s u] < \infty \wedge \|g(x, t)\| \le m_u(t) \text{ for } (\lambda, x, t) \in \Lambda \times \Omega \times [0, T]. \end{cases}$$

Then: any solution of (I) is in BV $[0, T]$ and left-continuous.

Sketch of proof:

$$r(t) := \int_0^t g(x(s), s)\, d\, u(s) \quad \text{for } t \in [0, T] \text{ and } \{\alpha_0, \dots, \alpha_m\} \text{ division of } [0, T]$$

$$\Rightarrow \quad \sum_{j=1}^m \|r(\alpha_j) - r(\alpha_{j-1})\| \le \sum_{j=1}^m \int_{\alpha_{j-1}}^{\alpha_j} \|g(x(s), s)\|\, d\,[\operatorname{var}_0^s u]$$

$$\le \int_0^T m_u\, d\,[\operatorname{var}_0^s u] < \infty, \quad \text{i.e. } r \in BV\,[0, T].$$

$$Dx = f(\lambda, x, t) + g(x,t)\,.\,Du, \quad x(0) = x(T), \tag{P}$$

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Theorem

Assume: (A) and (B). Then: (P) and (I) are equivalent.

Sketch of proof:

Let x be a solution of (P).

Then, by definition, $x \in G[0,T]$ is left-continuous, $x(t) \in \Omega$ for $t \in [0,T]$ and

$$D\big(x - F_\lambda(x)\big) = 0 \in \mathcal{D}^*,$$

where

$$F_\lambda(x)(t) = \int_0^t f(\lambda, x(s), s)\, ds + \int_0^t g(x(s), s)\, d\,u(s) \in \mathbb{R}^n \text{ for } \lambda \in \Lambda.$$

$x - F_\lambda(x) \in BV[0,T]$ is left-continuous for all $\lambda \in \Lambda \;\Rightarrow$

$$\exists\, c \in \mathbb{R}^n : \; x(t) - F_\lambda(x)(t) = c \text{ for all } \lambda \in \Lambda,\ t \in [0,T].$$

It follows that $x(0) = x(T) = c$ and x is a solution to (I). □

Put $\quad F(\lambda, x, t) = \int_0^t f(\lambda, x, s)\, ds + \int_0^t g(x, s)\, d\, u(s)$ for $(\lambda, x, t) \in \Lambda \times \Omega \times [0, T]$.

Proposition (Schwabik)

Assume: (A) and (B).

Then: for any $\lambda \in \Lambda$ there are

$$\begin{cases} \varkappa_\lambda : [0, T] \to \mathbb{R} & \text{nondecreasing and left-continuous,} \\ \omega_\lambda : [0, \infty) \to \mathbb{R} & \text{continuous, increasing, } \omega_\lambda(0) = 0 \end{cases}$$

such that $F(\lambda, \cdot, \cdot) \in \mathcal{F}(\Omega \times [0, T], \varkappa_\lambda, \omega_\lambda)$ for $\lambda \in \Lambda$, i.e.

$$\|F(\lambda, x, t_2) - F(\lambda, x, t_1)\| \leq |\varkappa_\lambda(t_2) - \varkappa_\lambda(t_1)|,$$

$$\|F(\lambda, x, t_2) - F(\lambda, x, t_1) - F(\lambda, y, t_2) + F(\lambda, y, t_1)\| \leq \omega(\|x - y\|)|\varkappa_\lambda(t_2) - \varkappa_\lambda(t_1)|$$

for $\lambda \in \Lambda,\ x, y \in \Omega,\ t_1, t_2 \in [0, T]$.

Moreover,

$$\int_0^t f(\lambda, x(r), r)\, d\, r + \int_0^t g(x(r), r)\, d\, u(r) = \int_0^t DF(\lambda, x(\tau), \sigma)$$

for $t \in [0, T], \lambda \in \Lambda$ and $x \in G[0, T]$ such that $x(s) \in \Omega$ for all $s \in [0, T]$.

$$\int_0^t DF(\lambda, x(\tau), \sigma) \approx \sum_P [F(\lambda, x(\tau_j), \sigma_j) - F(\lambda, x(\tau_j), \sigma_{j-1})],$$

where $P = \{\tau_j, [\sigma_{j-1}, \sigma_j]\}$ are tagged partitions of $[0, t]$.

$$x(t) = x_0 + \int_{t_0}^{t} DF(x(\tau), \sigma) \qquad \text{(K)}$$

Theorem (Kurzweil, Schwabik)

Assume:

- $F \in \mathcal{F}(\Omega \times [0, T], \varkappa, \omega)$,
- $(x_0, t_0) \in \Omega$ and $x_0 + F(x_0, t_0+) - F(x_0, t_0) \in \Omega$.

Then: (K) has a solution x on a neighborhood of t_0 such that $x(t_0) = x_0$.

$$F(\lambda, x, t) = \int_0^t f(\lambda, x, s)\, ds + \int_0^t g(x, s)\, d\, u(s) \Rightarrow$$

$$x(t) = x(T) + \int_0^t DF(\lambda, x(\tau), \sigma) \quad \equiv \quad \mathrm{(P)} \quad \equiv \quad \mathrm{(I)}.$$

$$x(t) = x(T) + \int_0^t DF(\lambda, x(\tau), \sigma) \qquad (\widetilde{P})$$

$$\widetilde{\Phi}(\lambda, x)(t) = x(T) + \int_0^t DF(\lambda, x(\tau), \sigma) \quad \text{for } \lambda \in \Lambda,\ x \in \overline{B(x_0, \rho)}, t \in [0, T].$$

Then
$$(\widetilde{P}) \Leftrightarrow x = \widetilde{\Phi}(x).$$

Definition (Krasnoselskii & Zabreiko, 1984 or Amann, 1990)

Let x_0 be a solution of $(\widetilde{P})$ for all $\lambda \in \Lambda$.

Then (λ_0, x_0) is a *bifurcation point* of $(\widetilde{P})$ if every its neighborhood in $\Lambda \times G[0, T]$ contains a solution (λ, x) of $(\widetilde{P})$ such that $x \neq x_0$.

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In other words, (λ_0, x_0) is a *bifurcation point* of $(\widetilde{P})$ if

$\exists$ sequence $\{(\lambda_n, x_n)\}$ of solutions to $(\widetilde{P})$ tending to (λ_0, x_0), while $x_n \neq x_0\ \forall\, n$.

$$x(t) = x(T) + \int_0^t DF(\lambda, x(\tau), \sigma) \quad \Leftrightarrow \quad x = \Phi(x, \lambda) \qquad (\widetilde{P})$$

$$\widetilde{\Phi}(\lambda, x)(t) = x(T) + \int_0^t DF(\lambda, x(\tau), \sigma)$$

Assumptions $(\widetilde{C})$

(i) $x_0 \equiv 0$ is a solution of (P) for all $\lambda \in \Lambda$ and $x(t) \in \Omega$ for all $x \in B(x_0, \rho) \subset G[0, T]$.

(ii) $F \in \mathcal{F}(\Omega \times [0, T], \varkappa, \omega)$ for each $\lambda \in \Lambda$.

(iii) There is $\gamma : [0, T] \to \mathbb{R}$ nondecreasing and such that for any $\varepsilon > 0$ we can find $\delta > 0$ such that

$$\|F(\lambda_1, x, t) - F(\lambda_2, x, t) - F(\lambda_1, x, s) + F(\lambda_2, x, s)\| < \varepsilon\,|\gamma(t) - \gamma(s)|$$

for $x \in \Omega$, $t, s \in [0, T]$ and $\lambda_1, \lambda_2 \in \Omega$ such that $|\lambda_1 - \lambda_2| < \delta$.

$(\widetilde{C}) \Rightarrow$ $\widetilde{\Phi}$ is continuous on $\Lambda \times B(x_0, \rho)$.

Theorem (Federson, Mawhin & C. Mesquita)

Assume: $(\widetilde{C})$ and there exist $[\lambda_1^*, \lambda_2^*] \subset \Lambda$ is such that

$$\mathrm{ind}_{LS}(\mathrm{Id} - \widetilde{\Phi}(\lambda_1^*, .), x_0) \neq \mathrm{ind}_{LS}(\mathrm{Id} - \widetilde{\Phi}(\lambda_2^*, .), x_0).$$

Then: there is $\lambda_0 \in [\lambda_1^*, \lambda_2^*]$ such that (λ_0, x_0) is a bifurcation point of $(\widetilde{P})$.

Existence of a bifurcation point

$$Dx = f(x,t) + g(\lambda, x, t)\,.\,Du, \quad x(0) = x(T), \tag{P}$$

$$\Phi(\lambda, x)(t) = x(T) + \int_0^t f(x(s), s)\,ds + \int_0^t g(\lambda, x(s), s)\,d\,u(s)$$

Assumptions (C)

(i) $x_0 \equiv 0$ is a solution of (P) for all $\lambda \in \Lambda$ and $x(t) \in \Omega$ for all $x \in B(x_0, \rho)$.

(ii) There is $\gamma : [0, T] \to \mathbb{R}$ nondecreasing and such that for any $\varepsilon > 0$ there is $\delta > 0$ such that

$$\Big\| \int_s^t [g(\lambda_2, x, r) - g(\lambda_1, x, r)]\,d\,\mathrm{var}_0^r\,u \Big\| < \varepsilon\,|\gamma(t) - \gamma(s)|$$

for $x \in \Omega$, $t, s \in [0, T]$ and $\lambda_1, \lambda_2 \in \Omega$ such that $|\lambda_1 - \lambda_2| < \delta$.

Corollary

Assume: (A), (B), (C) and there exist $[\lambda_1^*, \lambda_2^*] \subset \Lambda$ is such that

$$\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(\lambda_1^*, .), x_0) \neq \mathrm{ind}_{LS}(\mathrm{Id} - \Phi(\lambda_2^*, .), x_0).$$

Then: there is $\lambda_0 \in [\lambda_1^*, \lambda_2^*]$ such that (λ_0, x_0) is a bifurcation point of (P).

Definition (Leray-Schauder index)

Let X be a Banach space and $\Omega \subset X$ be open and bounded. Let $\Phi : \overline{\Omega} \to X$ be compact and let a be an isolated fixed point of Φ. Then

$$\mathrm{ind}_{LS}(\mathrm{Id} - \Phi, a) = \deg_{LS}[\mathrm{Id} - \Phi, B(a, r), 0] \quad \text{for small } r > 0.$$

Proposition

- (i) Let $\mathcal{L} : X \to X$ be a compact linear operator. Let $R : B(0, \rho) \to X$ a compact operator such that
$$\lim_{\|x\|_X \to 0} \frac{\|R(x)\|_X}{\|x\|_X} = 0.$$
If λ is not a characteristic value of $\mathcal{L}$, then
$$\mathrm{ind}_{LS}(\mathrm{Id} - \lambda\,\mathcal{L} - R, 0) = \mathrm{ind}_{LS}(\mathrm{Id} - \lambda\,\mathcal{L}, 0) = \pm 1.$$
- (ii) Let $\mathcal{L}_1, \mathcal{L}_2 : X \to X$ be compact linear operators such that $\mathrm{Id} - \mathcal{L}_1$ and $\mathrm{Id} - \mathcal{L}_2$ are invertible. Then
$$\mathrm{ind}_{LS}((\mathrm{Id} - \mathcal{L}_1)\,(\mathrm{Id} - \mathcal{L}_2), 0) = \mathrm{ind}_{LS}(\mathrm{Id} - \mathcal{L}_1, 0)\,\mathrm{ind}_{LS}(\mathrm{Id} - \mathcal{L}2, 0).$$
- (iii) Let $\mathcal{L} : X \to X$ be a compact linear operator. If λ_0 is a characteristic number of $\mathcal{L}$ of algebraic multiplicity α_0, then
$$\mathrm{ind}_{LS}(\mathrm{Id} - (\lambda_0 + \delta)\,\mathcal{L}, 0) = (-1)^{\alpha_0}\,\mathrm{ind}_{LS}(\mathrm{Id} - (\lambda_0 - \delta)\,\mathcal{L}, 0).$$
for all sufficiently small $\delta > 0$.

Example (Federson, Mawhin, Mesquita)

Let $b, c \in L^1[0,1]$, $\int_0^1 b(s)\,ds \neq 0$, $\lambda \in \Lambda - (-2,2)$. Consider impulsive problem

$$x' = \lambda\, b(t)\, x + c(t)\, x^2, \quad \Delta^+ x(\tfrac{1}{2}) = x^2(\tfrac{1}{2}), \quad x(0) = x(1), \tag{E1}$$

or equivalently

$$x(t) = x(1) + \lambda \int_0^t b(s)\, x(s)\, ds + \int_0^t c(s)\, x^2(s)\, ds + \int_0^t x^2(s)\, d\chi_{(\frac{1}{2},1]}(s)$$

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or equivalently

$$x(t) = \underbrace{x(1) + \lambda \overbrace{\int_0^t b(s)\, x(s)\, ds}^{(L\,x)(t)} + \overbrace{\int_0^t c(s)\, x^2(s)\, ds + \int_0^t x^2(s)\, d\chi_{(\frac{1}{2},1]}(s)}^{(R\,x)(t)}}_{\Phi(\lambda, x)(t)}$$

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- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.

$$(\mathcal{L}x)(t) = x(1) + (L\,x)(t) \implies (I - \mathcal{L})\, z = 0 \;\Leftrightarrow\; z' = b(t)\, z,\; z(0) = z(1) \;\Leftrightarrow\; z \equiv 0,$$

i.e. $\mathrm{Id} - \mathcal{L} : G[0,1] \to G[0,1]$ is invertible.

Let $b, c \in L^1[0,1]$, $\int_0^1 b(s)\,ds \neq 0$, $\lambda \in \Lambda = (-2,2)$. Consider impulsive problem

$$x' = \lambda\, b(t)\, x + c(t)\, x^2, \quad \Delta^+ x(\tfrac{1}{2}) = x^2(\tfrac{1}{2}), \quad x(0) = x(1), \tag{E1}$$

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- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.

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i.e. $\mathrm{Id} - \mathcal{L} : G[0,1] \to G[0,1]$ is invertible.

Hence, $\mathrm{Id} - \Phi(\lambda, x) = (\mathrm{Id} - \mathcal{L} - (\lambda - 1)\,L)\,x - R(x)$

$$= (\mathrm{Id} - \mathcal{L})\Big(\big[\mathrm{Id} - (\lambda - 1)\,(\mathrm{Id} - \mathcal{L})^{-1}\,L\big]\,x - (\mathrm{Id} - \mathcal{L})^{-1}\,R(x)\Big)$$

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- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.
- $(\mathcal{L}x)(t) = x(1) + (L\,x)(t)$
- $I - \Phi(\lambda, x) = [I - \mathcal{L} - (\lambda - 1)\,L]\,x - R(x) = (I - \mathcal{L})\left([I - (\lambda - 1)\,(I - \mathcal{L})^{-1}\,L]\,x - (I - \mathcal{L})^{-1}\,R(x)\right)$

Example (Federson, Mawhin, Mesquita)

$$x(t) = \underbrace{x(1) + \lambda \overbrace{\int_0^t b(s)\,x(s)\,ds}^{(L\,x)(t)} + \overbrace{\int_0^t c(s)\,x^2(s)\,ds + \int_0^t x^2(s)\,d\chi_{(\frac{1}{2},1]}(s)}^{(R\,x)(t)}}_{\Phi(\lambda,x)(t)} \tag{E1}$$

- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.
- $(\mathcal{L}x)(t) = x(1) + (L\,x)(t)$
- $I - \Phi(\lambda, x) = [I - \mathcal{L} - (\lambda - 1)\,L]\,x - R(x) = (I - \mathcal{L})\left([I - (\lambda-1)\,(I-\mathcal{L})^{-1}\,L]\,x - (I-\mathcal{L})^{-1}\,R(x)\right)$

Furthermore, $\lambda_0 = -1$ is the only characteristic value of $(\mathrm{Id} - \mathcal{L})^{-1}\,L$ and, as

$$z + (\mathrm{Id} - \mathcal{L})^{-1} L\,z = 0 \;\Leftrightarrow\; \underbrace{z(t) - (\mathcal{L}\,z)(t)}_{z(1)-(L\,z)(t)} + (L\,z)(t) \equiv 0 \;\Leftrightarrow\; z(t) \equiv z(1),$$

its multiplicity is 1. Thus, with respect to (iii), we have for each $\delta \in (0,1)$

$$\mathrm{ind}_{LS}(\mathrm{Id} - (-1+\delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0) = -\,\mathrm{ind}_{LS}(\mathrm{Id} - (-1-\delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0).$$

$$x(t) = \underbrace{x(1) + \lambda \overbrace{\int_0^t b(s)\,x(s)\,ds}^{(L\,x)(t)} + \overbrace{\int_0^t c(s)\,x^2(s)\,ds + \int_0^t x^2(s)\,d\chi_{(\frac{1}{2},1]}(s)}^{(R\,x)(t)}}_{\Phi(\lambda,x)(t)} \tag{E1}$$

- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.
- $(\mathcal{L}x)(t) = x(1) + (L\,x)(t)$
- $I - \Phi(\lambda, x) = [I - \mathcal{L} - (\lambda - 1)\,L]\,x - R(x) = (I - \mathcal{L})\,\left([I - (\lambda - 1)\,(I - \mathcal{L})^{-1}\,L]\;x - (I - \mathcal{L})^{-1}\,R(x)\right)$
- $\mathrm{ind}_{LS}(\mathrm{Id} - (-1 + \delta)\,(\mathrm{Id} - \mathcal{L})^{-1}L, 0) = -\,\mathrm{ind}_{LS}(\mathrm{Id} - (-1 - \delta)\,(\mathrm{Id} - \mathcal{L})^{-1}L, 0)$ if $\delta \in (0, 1)$.

$$x(t) = \underbrace{x(1) + \lambda \overbrace{\int_0^t b(s)\,x(s)\,ds}^{(L\,x)(t)} + \overbrace{\int_0^t c(s)\,x^2(s)\,ds + \int_0^t x^2(s)\,d\chi_{(\frac{1}{2},1]}(s)}^{(R\,x)(t)}}_{\Phi(\lambda,x)(t)} \tag{E1}$$

- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.
- $(\mathcal{L}x)(t) = x(1) + (L\,x)(t)$
- $I - \Phi(\lambda, x) = [I - \mathcal{L} - (\lambda - 1)\,L]\,x - R(x) = (I - \mathcal{L})\,\left([I - (\lambda - 1)\,(I - \mathcal{L})^{-1}\,L]\;x - (I - \mathcal{L})^{-1}\,R(x)\right)$
- $\mathrm{ind}_{LS}(\mathrm{Id} - (-1 + \delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0) = -\,\mathrm{ind}_{LS}(\mathrm{Id} - (-1 - \delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0)$ if $\delta \in (0,1)$.

Finally, as $\displaystyle\lim_{\|x\|_\infty \to 0} \frac{\|R(x)\|_\infty}{\|x\|_\infty} = 0$, by (i) and (ii), we get

$$\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(\delta,\cdot), 0) = \mathrm{ind}_{LS}((\mathrm{Id} - \mathcal{L})\,(\mathrm{Id} - (-1+\delta)\,L), 0)$$
$$= \mathrm{ind}_{LS}((\mathrm{Id} - \mathcal{L}, 0)\,\mathrm{ind}_{LS}((\mathrm{Id} - (-1+\delta)\,L), 0) = -\mathrm{ind}_{LS}((\mathrm{Id} - \mathcal{L}, 0)\,\mathrm{ind}_{LS}((\mathrm{Id} - (-1-\delta)\,L), 0)$$
$$= -\mathrm{ind}_{LS}((\mathrm{Id} - \mathcal{L})\,(\mathrm{Id} - (-1-\delta)\,L), 0) = -\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(-\delta,\cdot), 0).$$

$$x(t) = \underbrace{x(1) + \lambda \overbrace{\int_0^t b(s)\, x(s)\, ds}^{(L\,x)(t)} + \overbrace{\int_0^t c(s)\, x^2(s)\, ds + \int_0^t x^2(s)\, d\chi_{(\frac{1}{2},1]}(s)}^{(R\,x)(t)}}_{\Phi(\lambda,x)(t)} \qquad \text{(E1)}$$

- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.
- $(\mathcal{L}x)(t) = x(1) + (L\,x)(t)$
- $I - \Phi(\lambda, x) = [I - \mathcal{L} - (\lambda - 1)\, L]\, x - R(x) = (I - \mathcal{L})\ \left([I - (\lambda - 1)\,(I - \mathcal{L})^{-1}\, L]\ x - (I - \mathcal{L})^{-1}\, R(x)\right)$
- $\mathrm{ind}_{LS}(\mathrm{Id} - (-1+\delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0) = -\ \mathrm{ind}_{LS}(\mathrm{Id} - (-1-\delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0)$ if $\delta \in (0,1)$.
- $\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(-1+\delta, \cdot), 0) = -\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(-1-\delta, \cdot), 0)$ if $\delta \in (0,1)$.

Example (Federson, Mawhin, Mesquita)

$$x(t) = \underbrace{x(1) + \lambda \overbrace{\int_0^t b(s)\, x(s)\, ds}^{(L\,x)(t)} + \overbrace{\int_0^t c(s)\, x^2(s)\, ds + \int_0^t x^2(s)\, d\chi_{(\frac{1}{2},1]}(s)}^{(R\,x)(t)}}_{\Phi(\lambda,x)(t)} \qquad \text{(E1)}$$

- $x_0(t) \equiv 0$ is a solution of (E1) for all $\lambda \in \mathbb{R}$.
- $(\mathcal{L}x)(t) = x(1) + (L\,x)(t)$
- $I - \Phi(\lambda, x) = [I - \mathcal{L} - (\lambda - 1)\, L]\, x - R(x) = (I - \mathcal{L})\, \left([I - (\lambda - 1)\,(I - \mathcal{L})^{-1}\, L]\; x - (I - \mathcal{L})^{-1}\, R(x) \right)$
- $\mathrm{ind}_{LS}(\mathrm{Id} - (-1+\delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0) = -\,\mathrm{ind}_{LS}(\mathrm{Id} - (-1-\delta)\,(\mathrm{Id} - \mathcal{L})^{-1} L, 0)$ if $\delta \in (0,1)$.
- $\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(-1+\delta, \cdot), 0) = -\mathrm{ind}_{LS}(\mathrm{Id} - \Phi(-1-\delta, \cdot), 0)$ if $\delta \in (0,1)$.

To summarize: by [Federson,Mawhin & Mesquita]

$\exists\, \delta^* > 0$ s.t. for any $\delta \in (0, \delta^*)$ there is a bifurcation point $(\lambda_*, 0)$ of (E1) with $\lambda_* \in (-\delta, \delta)$.

$$Dx = f(\lambda, x, t) + g(x, t)\,.\,Du, \quad x(0) = x(T), \tag{P}$$

$$\Phi(\lambda, x)(t) = x(T) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,d\,u(s)$$

x_0 is a solution of (P) for all $\lambda \in \Lambda$ and $\rho > 0$ is such that $x(t) \in \Omega$ for all $x \in \overline{B(x_0, \rho)}$.

Theorem

Assume: for any $(\lambda, x) \in \Lambda \times \overline{B(x_0, \rho)}$, $\Phi(\lambda, \cdot)$ has a derivative $\Phi'_x(\lambda, x)$ continuous on $\Lambda \times \overline{B(x_0, \rho)}$ and such that

$\mathrm{Id} - \Phi'_x(\lambda_0, x_0)$ is an isomorphism of $G[0, T]$ onto $G[0, T]$.

Then: (λ_0, x_0) is not a bifurcation point of the problem (P).

Sketch of proof: We have $\Phi(\lambda, x_0) = x_0$ for all $\lambda \in \Lambda$. Abstract Implicit Function Theorem $\Rightarrow$

there are neighborhoods $\mathcal{V} \subset \Lambda$ of λ_0 and $\mathcal{W} \subset \overline{B(x_0, \rho)}$ of x_0 such that

for all $\lambda \in \mathcal{V}$ there is exactly one $x \in \mathcal{W}$ such that $x = \Phi(\lambda, x)$.

Thus, $x = x_0$ is the only function satisfying the relations

$$x = \Phi(\lambda, x) \quad \text{for any } \lambda \in \mathcal{V},$$

i.e. (λ_0, x_0) can not be a bifurcation point of $\Phi(\lambda, x) = x$. □

$$Dx = f(\lambda, x, t) + g(x, t)\,.\,Du, \quad x(0) = x(T), \tag{P}$$

$$\Phi(\lambda, x)(t) = x(T) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,d\,u(s)$$

x_0 is a solution of (P) for all $\lambda \in \Lambda$ and $\rho > 0$ is such that $x(t) \in \Omega$ for all $x \in \overline{B(x_0, \rho)}$.

Proposition

Assume: (A), (B), (C) and

(D) (i) $f'_x(\lambda, \cdot, \cdot)$ is Carathéodory on $\Omega \times [0, T]$ for $(\lambda, t \in \Lambda \times [0, T]$;

(ii) $g'_x(\cdot, t)$ is continuous on Ω for $t \in [0, T]$ and there is $\widetilde{m}_u$ such that

$$\|g'_x(x, t)\| \le \widetilde{m}_u(t) \text{ and } \int_0^T \widetilde{m}_u(s)\,d\,[\operatorname{var}_0^s u] < \infty \text{ for } (x, t) \in \Omega \times [0, T].$$

Then: $$\Big(\Phi'_x(\lambda, x_0)\,z\Big)(t) = z(T) + \int_0^t [f'_x(\lambda, x_0(\tau), \tau)]\,z(\tau)d\,\tau + \int_0^t [g'_x(x_0(\tau), \tau)]\,z(\tau)d\,u(\tau)$$

for $z \in G\,[0, T]$, $t \in [0, T]$ and $\lambda \in \Lambda$.

$$A_\lambda(t) = f'_x(\lambda, x_0(t), t), \quad B(t) = g'_x(x_0(t), t) \Rightarrow$$

$$[\mathrm{Id} - \widetilde{\Phi}'_x(\lambda, x_0)]\,z = 0 \quad \text{iff} \quad z(t) = z(T) + \int_0^t d\left[\int_0^s A_\lambda(\tau)\,d\tau + \int_0^s B(\tau)\,d\,u(\tau)\right] z(s)$$

$$Dz = [A_\lambda + B\,.\,Du]\,z, \quad z(0) = z(T)$$

$$Dx = f(\lambda, x, t) + g(x, t) \,.\, Dh, \quad x(0) = x(T), \tag{P}$$

$$\Phi(\lambda, x)(t) = x(T) + \int_0^t f(\lambda, x(s), s)\, ds + \int_0^t g(x(s), s)\, d\, u(s)$$

x_0 is a solution of (P) for all $\lambda \in \Lambda$ and $\rho > 0$ is such that $x(t) \in \Omega$ for all $x \in \overline{B(x_0, \rho)}$.

Theorem

Assume: (A), (B), (C), (D) and put

$$A_\lambda(t) := f'_x(\lambda, x_0(t), t) \;\text{ for }\; (\lambda, t) \in \Lambda \times [0, T],$$

$$B(t) := g'_x(x_0(t), t) \;\text{ for }\; t \in [0, T].$$

Then: (λ_0, x_0) is not a bifurcation point of (P) whenever the equation

$$z(t) = z(T) + \int_0^t d\left[\int_0^s A_\lambda(\tau)\, d\tau + \int_0^s B(\tau)\, d\, u(\tau)\right] z(s)$$

has only the trivial solution.

P r o o f relies on our assertions given above and on the Fredholm property of linear boundary value problems for GODEs.

Example 1

Consider again

$$x(t) = x(1) + \int_0^t \lambda\, b(s)\, x(s) + c(s)\, x^2(s)\, ds + \int_0^t c(s)\, x^2(s)\, d\chi_{(1/2,1]}(s) \qquad \text{(E1)}$$

where $b, c \in L^1[0,1]$ and $\int_0^1 b(s)\, ds \neq 0.$

$x_0(t) \equiv 0$ is a solution for all $\lambda \in \Lambda$ and the linearization at $x_0 \equiv 0$ is given by

$$z' = \lambda\, b(t)\, z,\ \ z(0) = z(1) \quad \Leftrightarrow \quad \begin{cases} \lambda = 0 \ \wedge \ z \equiv const, \\ \lambda \neq 0 \ \wedge \ z \equiv 0. \end{cases}$$

We know that there is $\delta^* > 0$ such that for any $\delta \in (0, \delta^*)$ we can find $\lambda_0 \in (-\delta, \delta)$ such that $(\lambda_0, 0)$ is the bifurcation point of (E1).

Our Theorem implies that

$(\lambda, 0)$ is not a bifurcation point of (Ex), if $\lambda \neq 0$.

Example 2

Proposition (A. Lomtatidze, 2016)

Assume: $q : [0, T] \to \mathbb{R}$ is continuous and such that

$$\int_0^T q_+(s)\,ds > 0, \quad 0 < \int_0^T q_-(s)\,ds < \frac{2}{\pi} \quad \text{and}$$

$$\int_0^T q_-(s)\,ds < \left(1 - \frac{\pi}{2}\int_0^T q_-(s)\,ds\right)\left(\int_0^T q_+(s)\,ds\right).$$

Then: for each $f \in L^1[0, T]$, the problem

$$y'' + q(t)\,y = f(t), \quad y(0) = y(T), \quad y'(0) = y'(T)$$

has exactly one solution y and the problem has a positive Green's function.

$$\left[\, q_+(t) := \max\{q(t), 0\}, \quad q_-(t) := -\min\{q(t), 0\} \,\right]$$

Example 2

It is known [Cid & Sanchez, 2020] that $y(t) = (2+\cos t)^3 =: y_0(t)$ is a solution to

$$y'' = (6.6-5.7\,\cos t-9\,\cos^2 t)\,y^{1/3}-0.3\,y^{2/3},\;\; y(0) = y(2\,\pi),\; y'(0) = y'(2\,\pi)$$

related to the Liebau valveless pumping phenomena.

Obviously, $y = y_0$ solves also for all $\lambda \in \mathbb{R}$ the impulsive problem

$$y'' = \lambda\,((2+\cos t)\,y' + 3\,(\sin t)\,y) + (6.6 - 5.7\,\cos t - 9\,\cos^2 t)\,y^{1/3} - 0.3\,y^{2/3},$$

$$\Delta^+ y(\pi) = 2\,(y(\pi))^3 - (y(\pi))^2 - 4\,y(\pi) + 3, \quad y(0) = y(2\pi),\; y'(0) = y'(2\pi),$$

or equivalently

$$x(t) = x(2\pi) + \int_0^t f(\lambda, x(s), s)\,ds + \int_0^t g(x(s), s)\,d\,u(s), \tag{E2}$$

where

$$x_1 = y,\;\; x_2 = y',\; x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad g(x,t) = \begin{pmatrix} 2\,x_1^3 - x_1^2 - 4\,x_1 + 3 \\ 0 \end{pmatrix}, \quad u(t) = \chi_{(\pi,2\pi]}(t),$$

$$f(\lambda, x, t) = \begin{pmatrix} x_2 \\ \lambda\,((2+\cos t)\,x_2 + 3\,(\sin t)\,x_1) + R(t)\,x_1^{1/3} - 0.3\,x_1^{2/3} \end{pmatrix},$$

$$R(t) = 6.6 - 5.7\,\cos t - 9\,\cos^2 t.$$

$$x(t) = x(2\pi) + \int_0^t f(\lambda, x(s), s)\, ds + \int_0^t g(x(s), s)\, d\, u(s), \tag{E2}$$

$x_0 = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$ is a solution to for all $\lambda \in \mathbb{R}$.

Linearization of (E2) around x_0 yields

$$z(t) = z(2\pi) + \int_0^t f_x'(\lambda, x_0(r), r)\, z(r)\, dr + g_x'(x_0(\pi), \pi)\, z(\pi)\, \chi_{(0,\pi]}(t) \quad \text{for} \quad t \in [0, 2\pi]. \tag{L}$$

Inserting $\lambda = 0$ and $x_0 = \begin{pmatrix} y_0 \\ y_0' \end{pmatrix}$ into (L), we get

$$z'' = q(t)\, z, \quad z(0) = z(2\pi),\ z'(0) = z'(2\pi), \tag{L_0}$$

where

$$q(t) = \frac{3\,(6 - 7\,\cos t - 10\,\cos^2 t)}{10\,(2 + \cos t)^2}.$$

Example 2

$$z'' = q(t)\, z,\ \ z(0) = z(2\pi),\ z'(0) = z'(2\pi) \quad \left[q(t) = \frac{3\,(6 - 7\,\cos t - 10\,\cos^2 t)}{10\,(2+\cos t)^2}\right]. \qquad (L_0)$$

Using Mathematica system we get:

$$0 < 1 - \frac{\pi}{2}\int_0^{2\pi} q_-(s)\, ds \approx 0.193328$$

and

$$\int_0^{2\pi} q_+(s)\, ds = \frac{1}{15}\,\left((59\sqrt{3} - 60)\,\pi - 2\sqrt{3}\,(6 + \arctan 1/3)\right) \approx 3.06682,$$

$$\int_0^{2\pi} q_-(s)\, ds \approx 0.513543 < \left(1 - \frac{\pi}{2}\int_0^{2\pi} q_-(s)\, ds\right)\left(\int_0^{2\pi} q_+(s)\, ds\right) \approx 0.592902.$$

Hence, by Lomtatidze, linear problem (L_0) has only the trivial solution, i.e. $\Phi'_x(0, x_0)$ is an isomorphism, and we can conclude:

There is a $\delta > 0$ such that $|\lambda| + \|x - x_0\|_\infty < \delta \implies (\lambda, x)$ is not a bifurcation point of (E2).

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