Multiple solutions of Dirichlet boundary value problems in billiard spaces

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Overview

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- authors investigate the paths of solutions, periodic paths, structure, presence of chaos, ...
- billiards with uneven surface: questions of existence and multiplicity of paths

Billiards

- billiards with uneven surface

[Gabor2016]

Gabor, G.: On the Dirichlet problem in billiard spaces, J. Math. Anal. Appl. 440 (2016) 677–691.

Dirichlet problem in multidimensional case

$$
\begin{aligned}
x''(t) &= f(t, x(t)), && \text{for a.e.} t \in [0, T],\ x(t) \in \operatorname{int} K,\\
\triangle x'(s) &= I(x(s), x'(s)), && \text{if } x(s) \in \partial K,\\
x(0) &= x(T) = 0.
\end{aligned}
$$

Billiards

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- impact law for absolutely elastic impacts (equal angles before and after a collision), i.e. in 1-dim. case:

$$x'(s+) = -x'(s-), \quad \text{if } |x(s)| = r,$$

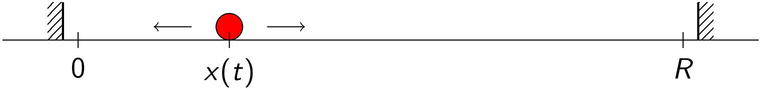
- one dimensional case: $K = [-r, r] \subset \mathbb{R}$,
- multidimensional case: $K \subset \mathbb{R}^{2k}$ has "smooth" boundary, $0 \in \operatorname{int} K$, K is strongly star-shaped,
- there exist infinitely many solutions.
- Proofs: based on shooting method and continuous dependence on initial conditions $x(0) = 0$, $\dot{x}(0) = d$.

Billiards

- billiards with uneven surface
- in 1D, there is no need of Lipschitz continuity
- there could be said more precise information about the infinite sequence of solutions
- using Schauder Fixed Point Theorem we are able to prove the existence of solutions with prescribed number of impacts
- I have used the transform into nonimpulsive (but singular) problem
- the idea is very simple and well known:

1D billiard Dirichlet problem – basic idea

- the ball moves inside of a line segment between 0 and R.



- uniform linear motion inside of the segment:

$$x''(t) = 0 \quad \text{if } x(t) \in (0, R)$$

- absolutely elastic bounce at the boundary:

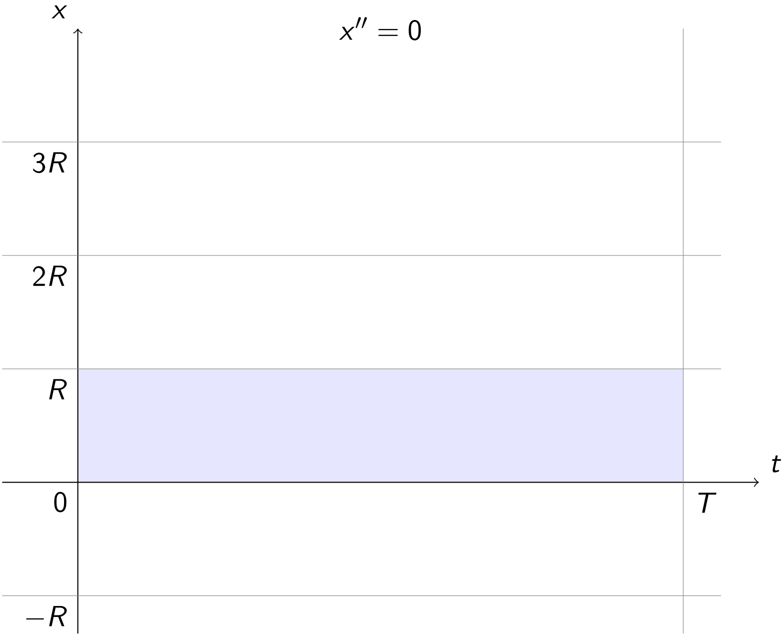
$$x'(t+) = -x'(t-) \quad \text{if } x(t) \in \{0, R\}.$$

1D billiard Dirichlet problem – basic idea

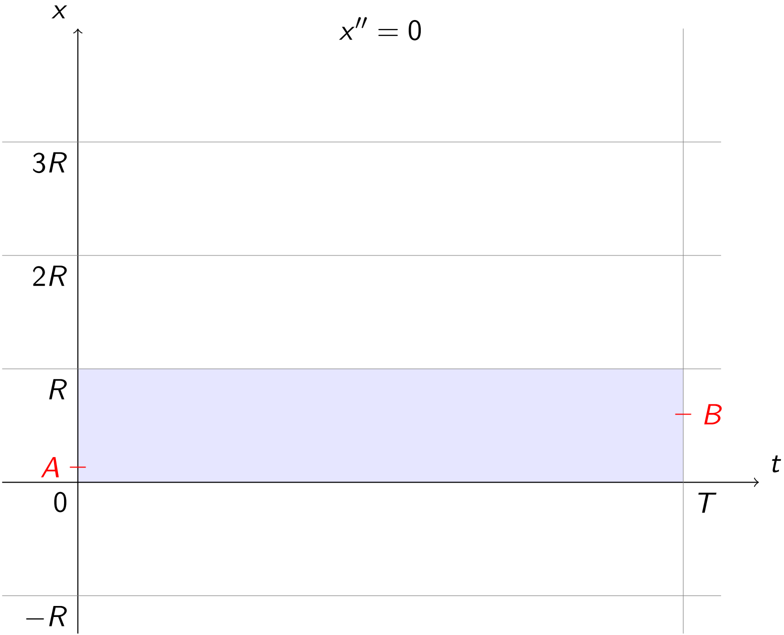
Dirichlet problem:

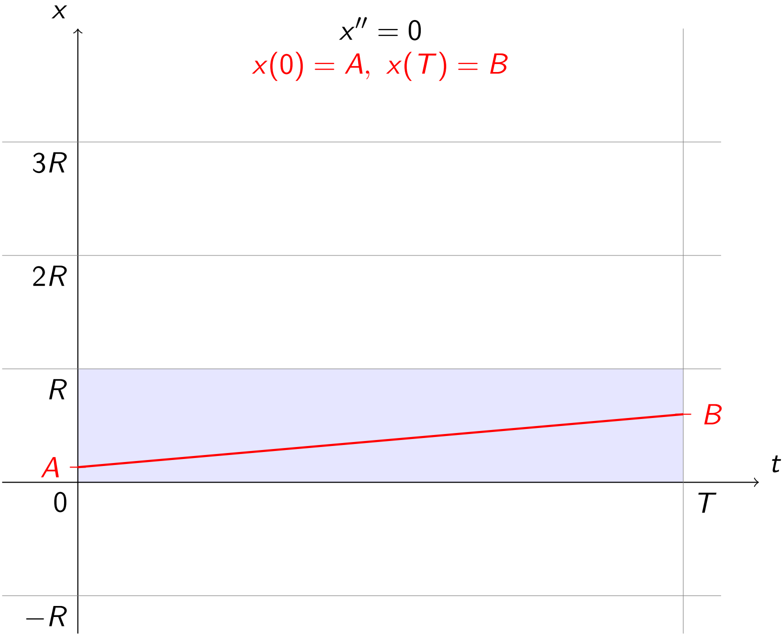
- positions $A, B \in (0, R)$ and time instant $T > 0$ are given
- we seek for solutions in 1D-billiard satisfying

$$x(0) = A, \quad x(T) = B.$$

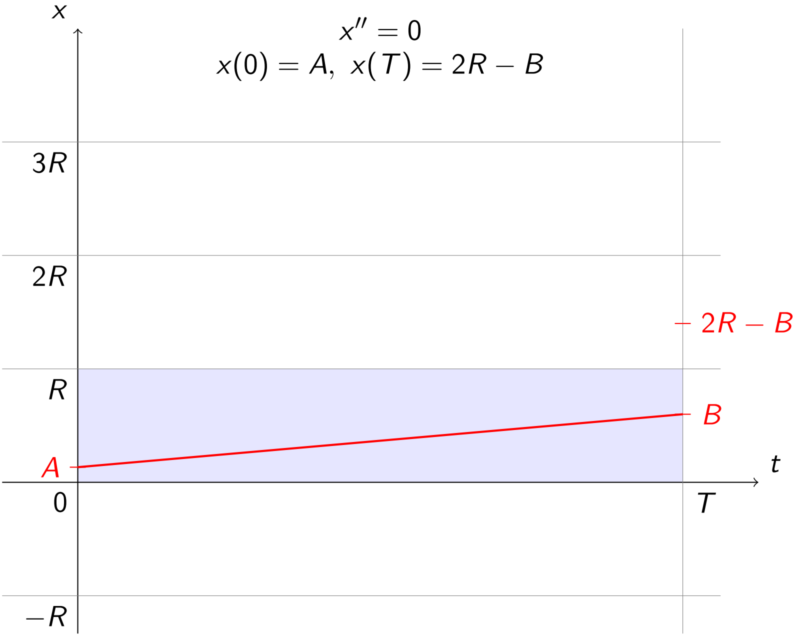


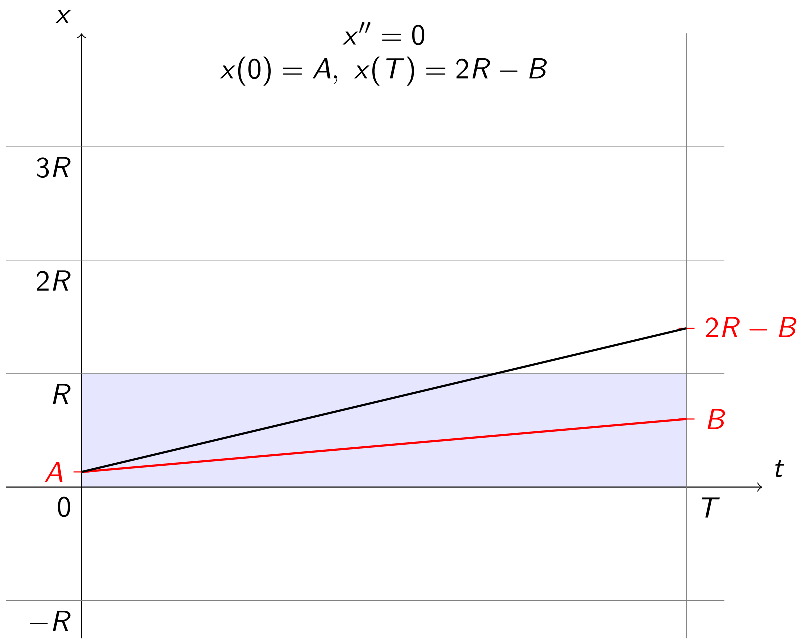
x
$x'' = 0$
3R
2R
R
t
0
T
−R


$x'' = 0$
x
t
$3R$
$2R$
R
0
$-R$
T
A
B

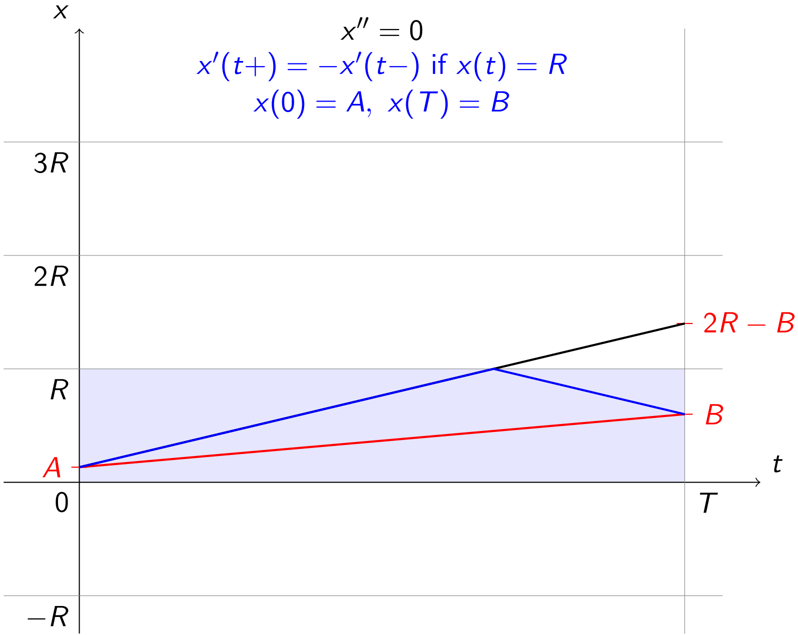


x
x'' = 0
x(0) = A, x(T) = B
3R
2R
R
B
A
t
0
T
−R

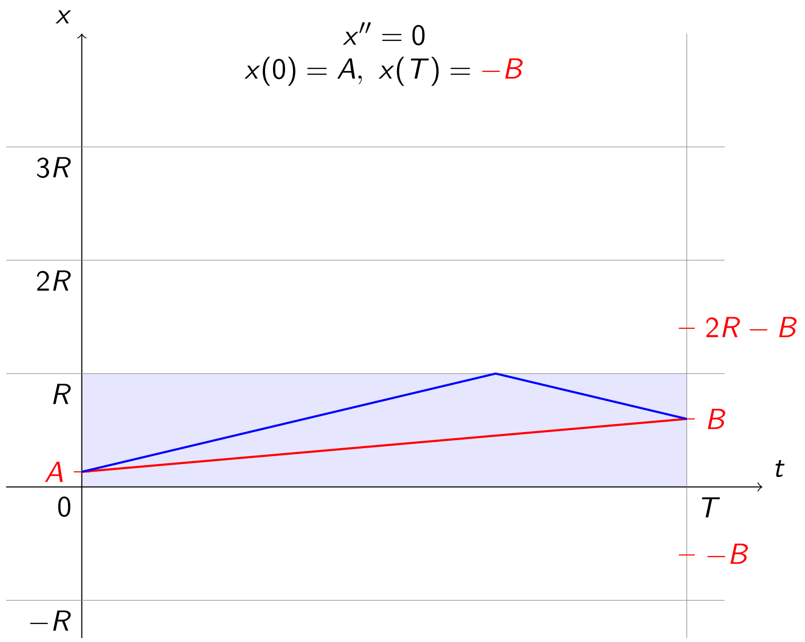

$x'' = 0$
$x(0) = A,\ x(T) = 2R - B$
x
t
$3R$
$2R$
R
0
$-R$
T
A
B
$2R - B$



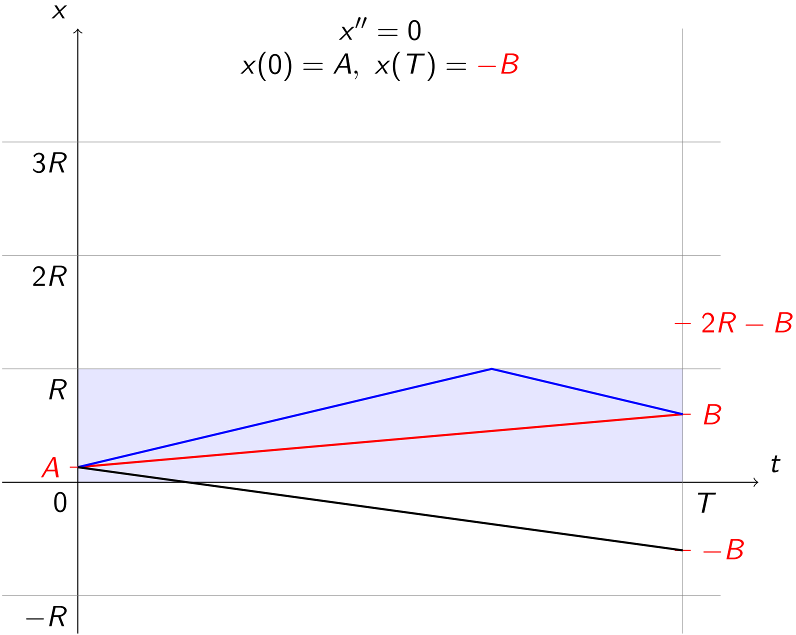
x
x'' = 0
x(0) = A, x(T) = 2R − B
3R
2R
R
A
0
−R
2R − B
B
t
T



x
x'' = 0
x'(t+) = −x'(t−) if x(t) = R
x(0) = A, x(T) = B
3R
2R
R
A
0
−R
2R − B
B
t
T



$x'' = 0$
$x(0) = A,\ x(T) = -B$
x
$3R$
$2R$
R
A
0
$-R$
$2R - B$
B
t
T
$-B$



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$x(0) = A,\ x(T) = -B$
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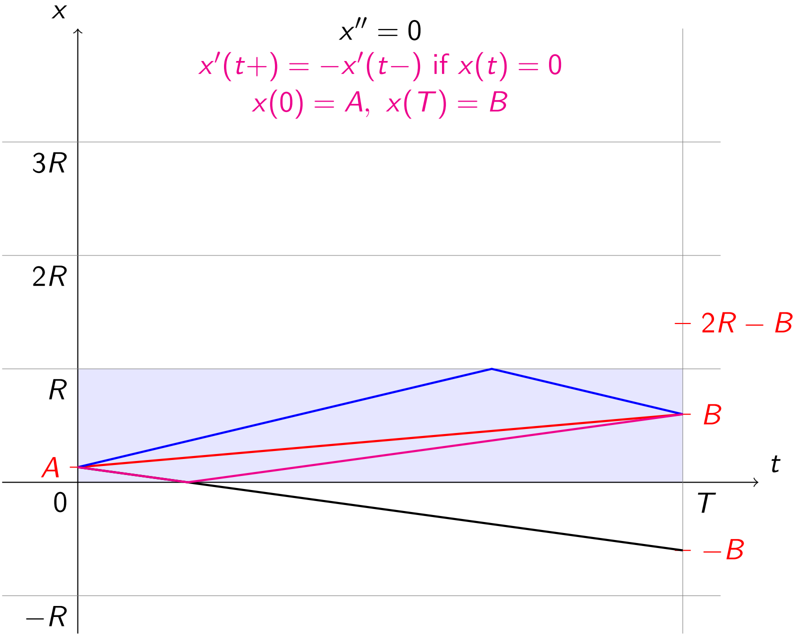
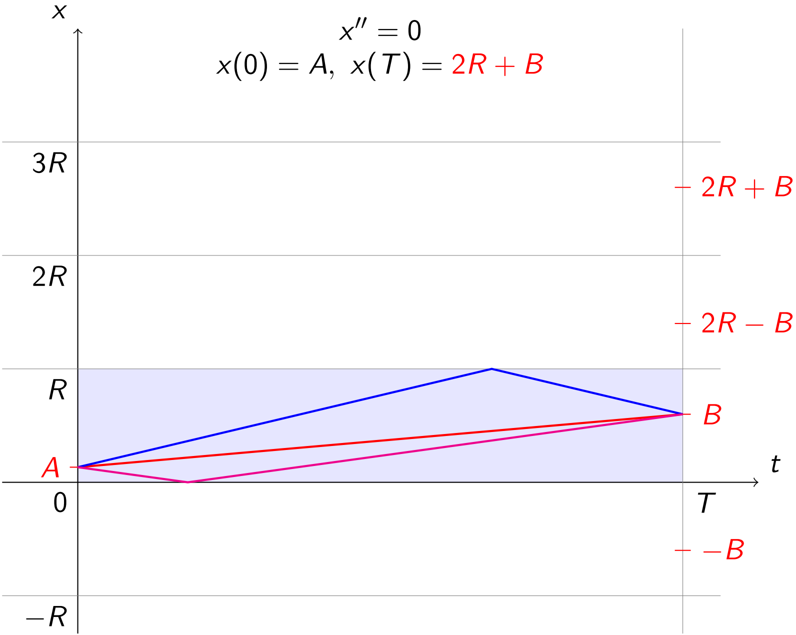
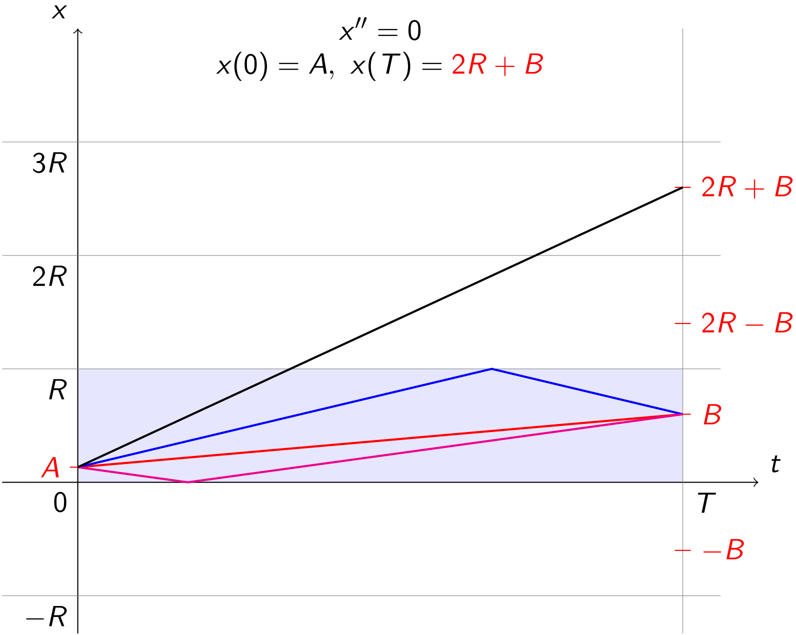
t
$3R$
$2R$
R
0
$-R$
A
T
$2R - B$
B
$-B$



x
x'' = 0
x'(t+) = −x'(t−) if x(t) = 0
x(0) = A, x(T) = B
3R
2R
2R − B
R
B
A
t
0
T
−B
−R



$x'' = 0$
$x(0) = A,\ x(T) = 2R + B$
x
$3R$
$2R$
R
0
$-R$
A
t
T
$2R + B$
$2R - B$
B
$-B$



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0
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T
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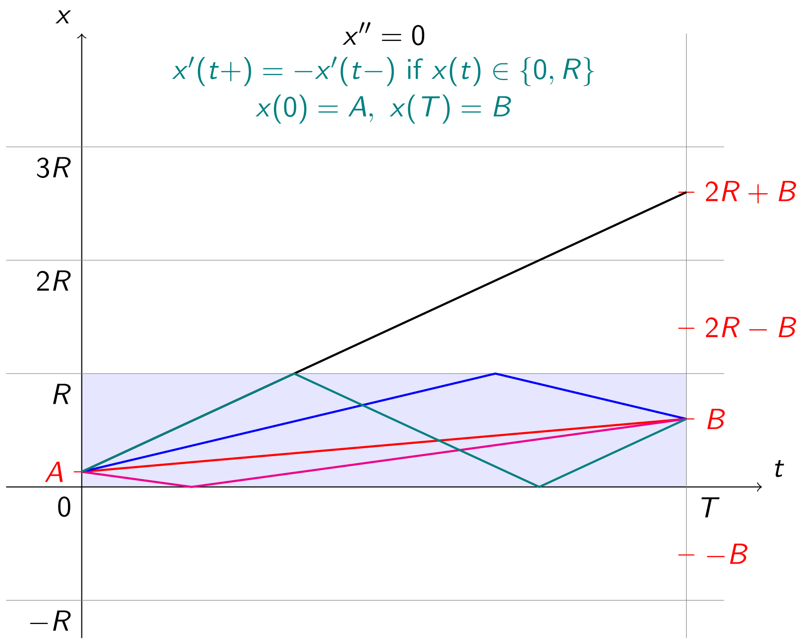
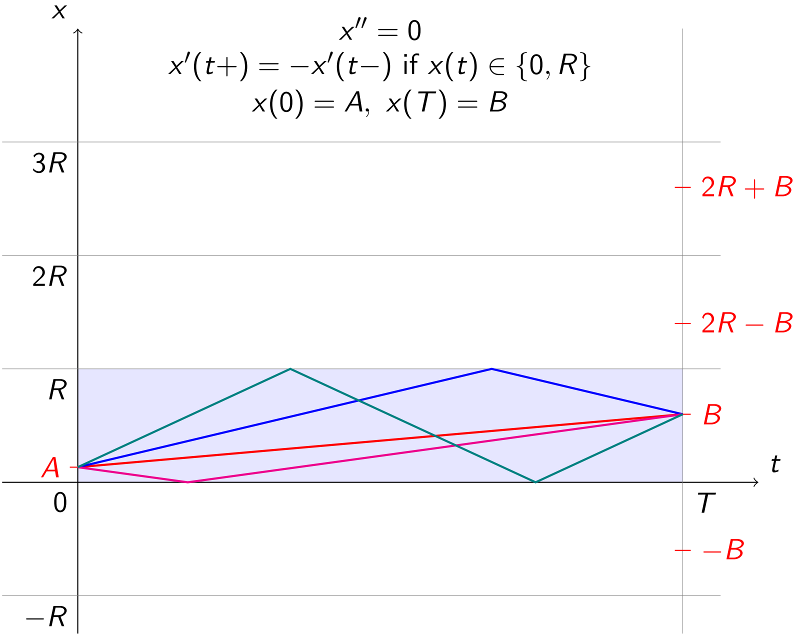
B
$-B$



x
$x'' = 0$
$x'(t+) = -x'(t-)$ if $x(t) \in \{0, R\}$
$x(0) = A,\ x(T) = B$
$3R$
$2R$
R
A
0
$-R$
$2R + B$
$2R - B$
B
$-B$
t
T



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x(0) = A, x(T) = B
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1D billiard Dirichlet problem – basic idea

Therefore the problem

(classicalBIL)

$$x''(t) = 0 \quad \text{if } x(t) \in (0, R),$$

$$x'(t+) = -x'(t-) \quad \text{if } x(t) \in \{0, R\},$$

$$x(0) = A,\ x(T) = B \quad (A, B \in (0, R))$$

- has infinitely many solutions,
- in particular: for each $p \in \mathbb{N}$ there exist exactly two solutions having exactly p impacts with the boundary.

1D billiard Dirichlet problem – basic idea

Therefore the problem

(classicalBIL)

$$
\begin{gathered}
x''(t) = 0 \quad \text{if } x(t) \in (0, R), \\
x'(t+) = -x'(t-) \quad \text{if } x(t) \in \{0, R\}, \\
x(0) = A,\ x(T) = B \quad (A, B \in (0, R))
\end{gathered}
$$

- has infinitely many solutions,
- in particular: for each $p \in \mathbb{N}$ there exist exactly two solutions having exactly p impacts with the boundary.
- can we generalize this idea for billiard with uneven billiard or subjected to external forces?

Dirichlet problem in billiard spaces

[Tomecek2019]

J.T.: Multiple solutions of a Dirichlet problem in one-dimensional billiard space, *Math. Notes (Miskolc)*, **20**(2):1261–1272, 2019.

[GaborTomecek2023]

Grzegorz Gabor and J.T.: Multiple solutions of the Dirichlet problem in multidimensional billiard spaces, J Fix Point Theory A **25**, 2023, Article No 7.

[RodriguezLopezTomecek2024]

Jorge Rodríguez-López and J. T.: Second–order discontinuous ODEs and billiard problems. J. Math. Anal. Appl., 536(1):128237, 2024.

Dirichlet problem in billiard spaces

[Gabor2023]

Gabor, G., Tessellation technique in solving the two-point boundary value problem in multidimensional billiard spaces, J. Math. Anal. Appl. **526**:127208,2023

1D billiard problem with velocity dependent RHS

- we investigate Dirichlet problem

(BIL)

$$\begin{aligned} &x'' = f(t, x, x') \quad \text{if } x(t) \in (0, R), \\ &x'(t+) = -x'(t-) \quad \text{if } x(t) \in \{0, R\}, \\ &x(0) = A, \; x(T) = B, \quad (A, B \in (0, R)) \end{aligned}$$

with $f \in Car([0, T] \times [0, R] \times \mathbb{R})$.

[Krajščáková–Tomeček,202?]

Věra Krajščáková, J.T.: Dirichlet problem in one-dimensinal billiard space with velocity dependent right-hand side, submitted.

1D billiard problem with velocity dependent RHS

Theorem

Let $A, B \in (0, R)$, $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$ and there exist $m \in \mathrm{L}^1([0, T])$ and nonnegative, increasing $\varphi \in \mathbb{C}([0, \infty))$ such that

$$|f(t, x, y)| \leq m(t) + \varphi(|y|) \qquad (*)$$

for a.e. $t \in [0, T]$, all $x \in [0, R]$, $y \in \mathbb{R}$.
If there exist $p \in \mathbb{N}$ and $L > 0$ such that

$$\Psi_1(L) = \frac{T}{R}(\overline{m} + T\varphi(L)) + 1 \leq p \leq \frac{T}{R}(L - \overline{m} - T\varphi(L)) - 1 = \Psi_2(L),$$

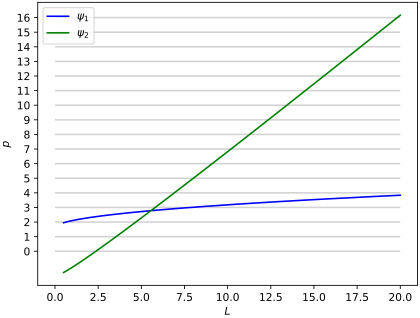
where $\overline{m} = \int_0^T m(t)\,\mathrm{d}t$, then there exist at least two solutions of (BIL) having exactly p impacts with the boundary.

Example

$$x'' = t^{\alpha} + \lambda |x|^{\beta} \operatorname{sgn} x + \omega |x'|^{\gamma} \operatorname{sgn} x',$$

for

$$T = 1,\ R = 1,\ \alpha = 1,\ \beta = 0.5,\ \gamma = 0.5,\ \omega = 0.5 \text{ and } \lambda = 0.1$$



1D billiard problem with velocity dependent RHS

Corollary

Let $A, B \in (0, R)$, $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$. Let $m \in \mathrm{L}^1([0, T])$ be such that

$$|f(t, x, y)| \leq m(t)$$

for a.e. $t \in [0, T]$, all $x \in [0, R]$, $y \in \mathbb{R}$, then for each $p \in \mathbb{N}$ satisfying

$$p \geq \frac{T}{R}\overline{m} + 1,$$

where $\overline{m} = \int_0^T m(t)\,\mathrm{d}t$, there exist at least two solutions of (BIL) having exactly p impacts with the boundary.

... for $\varphi \equiv 0$; L can be arbitrarily large ...

1D billiard problem with velocity dependent RHS

Theorem

Let $A, B \in (0, R)$, $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$ satisfy (*) and

$$\limsup_{y \to \infty} \frac{\varphi(y)}{y} < \frac{1}{2T}.$$

Then (BIL) has infinitely many solutions. In particular, there exists an increasing sequence of positive integers $\{p_n\}$ such that for each $n \in \mathbb{N}$ there exists at least two solutions of (BIL) with exactly p_n impacts with the boundary.

... satisfied for sublinear φ; or linear with small linear coefficient

The proofs

- Original problem (BIL):
  - problem on $[0, T] \times [0, R] \times \mathbb{R}$,
  - regular RHS,
  - problem with (state-dependent) impulses.

Original problem (BIL)

(BIL)

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&x(0) = A,\ x(T) = B, \quad (A, B \in (0, R))
\end{aligned}
$$

where $f \in \mathrm{Car}([0, T] \times [0, R] \times \mathbb{R})$.

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- Extended problems (EXT):
  - problem on $[0, T] \times \mathbb{R} \times \mathbb{R}$,
  - non-impulsive problem,
  - singular RHS; possibly unbounded in the last variable.

Construction of (EXT)

- New right-hand side of ODE,
- the extension of f:

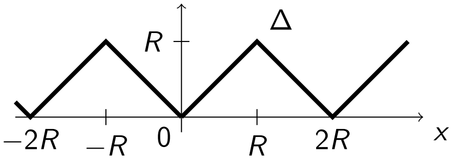
$$f^*(t,x,y) = \begin{cases} f(t, x - 2kR, y) \\ \qquad \text{if } x \in (2kR, (2k+1)R), \\ -f(t, 2(k+1)R - x, -y) \\ \qquad \text{if } x \in ((2k+1)R, 2(k+1)R), \\ 0, \qquad \text{if } x = kR,\ k \in \mathbb{Z} \end{cases}$$

- odd, $2R$-periodic in the second variable

Construction of (EXT)

- We define a folding function Δ:

$$\Delta(s) = \begin{cases} s - 2kR & \text{if } s \in [2kR, (2k+1)R), \\ 2(k+1)R - s & \text{if } s \in [(2k+1)R, 2(k+1)R). \end{cases}$$



- $\Delta : \mathbb{R} \to [0, R]$, idempotent, $2R$-periodic, continuous.

We consider these extended problems

(EXT)

$$y'' = f^*(t, y, y')$$

$$y(0) = A, \quad y(T) = \tilde{B}$$

where

$$\Delta(\tilde{B}) = B,$$

i.e.

$$\tilde{B} = B,\ 2R - B,\ 2R + B,\ \dots,\ -B,\ -2R + B,\ \dots$$

Lemma: (EXT) $\to$ (BIL)

If $y \in (EXT)$ is strictly monotone, then

$$\Delta \circ y$$

solves (BIL) having exactly

$$\left| \left\lfloor \frac{A}{R} \right\rfloor - \left\lfloor \frac{\tilde{B}}{R} \right\rfloor \right|$$

impacts with the boundary.

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solves (BIL) having exactly

$$\left| \left\lfloor \frac{A}{R} \right\rfloor - \left\lfloor \frac{\tilde{B}}{R} \right\rfloor \right|$$

impacts with the boundary.

Question: Does a strictly monotone solution of (EXT) exist? (moreover, we want at least two)

- Spoiler: Yes, if $\tilde{B}$ is in an appropriate distance from A.
- But, first we define two more auxiliary problems.

The proofs

- Original problem (BIL):
  - problem on $[0, T] \times [0, R] \times \mathbb{R}$,
  - regular RHS,
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  - problem on $[0, T] \times \mathbb{R} \times \mathbb{R}$,
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- Truncated problems in the last variable (TRUNC):
  - class of problems on $[0, T] \times \mathbb{R} \times \mathbb{R}$
  - bounded in the last variable,
  - depending on a real parameter L
  - but still singular RHS.

Construction of (TRUNC)

- Truncation of f^* in the last variable: for each $L > 0$ we define

$$f_L^*(t, x, y) = f^*(t, x, \max\{-L, \min\{y, L\}\})$$

- f_L^* is bounded by a Lebesgue integrable function over $[0, T]$
- it is still singular in x

(TRUNC)

$$y'' = f_L^*(t, y, y').$$
$$y(0) = A, \quad y(T) = \tilde{B}$$

$$y \in (\text{TRUNC}) \ \wedge \ |y'| \leq L \quad \Longrightarrow \quad y \in (\text{EXT})$$
$$y \in (\text{TRUNC}) \ \wedge \ |y'| \leq L \ \wedge \ y' \neq 0 \quad \Longrightarrow \quad \Delta \circ y \in (\text{BIL})$$

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  - but still singular RHS.
- Regularized problems (REG(n)):
  - sequence of regular problems on $[0, T] \times \mathbb{R} \times \mathbb{R}$,
  - non-impulsive problem,
  - regular RHS.

Construction of (REG(n))

- $f_L^*(t, x, y)$ has possible discontinuity at $x = kR$ (each $k \in \mathbb{Z}$)

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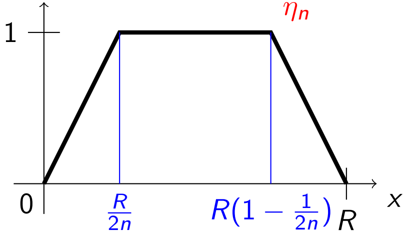
- $f_L^*(t, x, y)$ has possible discontinuity at $x = kR$ (each $k \in \mathbb{Z}$)
- we define regularizing functions $\eta_n : \mathbb{R} \to \mathbb{R}$



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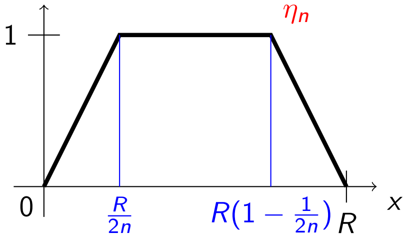

- R-periodic; continuous; zero at kR, $k \in \mathbb{Z}$; $\eta_n \to 1$ on $(0, R)$.

Sequence of regular problems

For $n \in \mathbb{N}$ we consider

(REG(n))

$$z'' = \eta_n(z) f_L^*(t, z, z'),$$
$$z(0) = A, \quad z(T) = \tilde{B}.$$

- regular RHS

Sequence of regular problems

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- regular RHS
- for each $A, \tilde{B} \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists at least one solution of (REG(n)), z_n ($\Leftarrow$ Schauder FPT).

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- regular RHS
- for each $A, \tilde{B} \in \mathbb{R}$ and $n \in \mathbb{N}$ there exists at least one solution of (REG(n)), z_n ($\Leftarrow$ Schauder FPT).
- $$\frac{\tilde{B} - A}{T} - \overline{m} - T\varphi(L) \le z_n'(t) \le \frac{\tilde{B} - A}{T} + \overline{m} + T\varphi(L),$$

  $t \in [0, T]$, $n \in \mathbb{N}$.

Sequence of regular problems

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- if $-L \le \frac{\tilde{B}-A}{T} - \overline{m} - T\varphi(L)$ & $\frac{\tilde{B}-A}{T} + \overline{m} + T\varphi(L) < 0$,
  then $z \in$ (TRUNC) & $-L \le z' < 0 \Rightarrow \Delta \circ z \in$ (BIL) (!!!)

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