On Bounded Monotone Solutions to Singular in the Time Variable Two-Dimensional Differential Systems

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1. Introduction

In the present report, for singular in the time variable differential systems

$$u_i' = f_i(t, u_1, u_2) \quad (i = 1, 2), \tag{1.1}$$

$$u_i' = g_i(t, u_{3-i}) \quad (i = 1, 2), \tag{1.1'}$$

we consider the problem on the existence of so-called Kneser solution, defined on the positive half-axis, and satisfying the initial condition

$$\lim_{t \to 0} u_1(t) = c. \tag{1.2}$$

Conditions that are unimprovable in a certain sense are given, guaranteeing, respectively, the existence, uniqueness, monotonicity, boundedness, and vanishness at infinity of a solution to this problem.

We use the following notation.

$$\mathbb{R} = \{-\infty, +\infty\}, \quad \mathbb{R}_+ = [0, +\infty[\,, \quad \mathbb{R}_- =]-\infty, 0];$$

$L(I)$ is the space of Lebesgue integrable real functions, defined in the interval I;

$L_{loc}(I)$, where I is either open or semi-open interval, is the space of defined in I real functions whose restrictions to any closed bounded interval contained in I are Lebesgue integrable;

We say that the function $f : I \times \mathbb{R}^m \to \mathbb{R}$ belongs to the Carathéodory space $K_{loc}(I \times \mathbb{R}^m)$ if the function $f(\cdot, x_1, \ldots, x_m) : I \to \mathbb{R}$ is measurable for any arbitrarily fixed $(x_1, \dots, x_m) \in \mathbb{R}^m$, the function $f(t, \cdot, \dots, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is continuous for almost all arbitrarily fixed $t \in I$, and

$$\max\Big\{|f(\cdot, x_1, \dots, x_m)| : \sum_{i=1}^{m} |x_i| \le x\Big\} \in L_{loc}(I)$$

for every positive constant x.

We investigate systems (1.1) and $(1.1')$ in the case, where

$$f_1 \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^2), \quad f_2 \in K_{loc}(\,]0, +\infty[\, \times \mathbb{R}^2), \tag{1.3}$$

and the conditions

$$f_i(t, 0, 0) = 0, \quad (-1)^i f_i(t, x_1, x_2) \ge 0 \tag{1.4}$$
$$\text{for } \ t > 0, \ x_1 \in \mathbb{R}_+, \ x_2 \in \mathbb{R}_- \ (i = 1, 2),$$

$$g_1 \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}), \quad g_2 \in K_{loc}(\,]0, +\infty[\, \times \mathbb{R}), \tag{1.5}$$

$$g_i(t, y) \ge g_i(t, x), \quad g_i(t, -x) = -g_i(t, x) \tag{1.6}$$
$$\text{for } \quad t > 0, \quad y \ge x \ \ (i = 1, 2)$$

are satisfied. At the same time, as noted above, we do not exclude cases where the differential systems under consideration have non-integrable singularities in the time variable at the point $t = 0$, namely, the cases where

$$\int_0^1 f_2(t, x_1, x_2)\, dt = +\infty, \quad \int_0^1 g_2(t, x_1)\, dt = +\infty \quad \text{for} \quad x_1 > 0, \ x_2 < 0.$$

Definition 1.1. A vector-function $(u_1, u_2) :]0, +\infty[\to \mathbb{R}^2$ is said to be a solution to system (1.1) if it is absolutely continuous on every closed bounded interval contained in $]0, +\infty[$, and satisfies this system almost everywhere in $]0, +\infty[$.

Definition 1.2. A solution (u_1, u_2) to system (1.1) is said to be:

1) **Kneser solution** if either

$$u_1(t) \geq 0, \quad u_2(t) \leq 0 \quad \text{for} \quad t > 0,$$

or

$$u_1(t) \leq 0, \quad u_2(t) \geq 0 \quad \text{for} \quad t > 0;$$

2) **strongly Kneser solution** if

$$u_1(t) u_2(t) < 0 \quad \text{for} \quad t > 0;$$

3) **monotone solution** if u_1 and u_2 are monotone functions;

4) **vanishing at infinity** if

$$\lim_{t \to +\infty} u_i(t) = 0 \ \ (i = 1, 2).$$

Before moving on to the formulation of the main results, let us briefly discuss the history of the problem.

Consider the linear homogeneous differential equation

$$v'' = p(t)v, \tag{1.7}$$

where $p \in L_{loc}(\mathbb{R}_+)$ is a nonnegative function. Let v_1 be a solution to this equation, satisfying the initial conditions

$$v_1(0) = 0, \quad v_1'(0) = 1,$$

and

$$v_2(t) = v_1(t) \int\limits_t^{+\infty} \frac{ds}{v_1^2(s)} \quad \text{for} \quad t > 0.$$

Then v_1, v_2 is a fundamental system of solutions to (1.7). On the other hand, it is evident that

$$v_1(t) \geq t, \quad 0 < v_2(t) \leq 1 \quad \text{for} \quad t > 0.$$

Therefore, in this case, the set of bounded solutions to (1.7) is a one-dimensional linear space with the basis v_2.

A. Kneser in [9] proved that if $f \in K_{loc}(\mathbb{R}_+ \times \mathbb{R})$ is a nondecreasing in the second argument function and $f(t, 0) \equiv 0$, then the nonlinear differential equation

$$v'' = f(t, v) \tag{1.8}$$

has the property analogous to that of equation (1.7). More precisely, no matter how a real number c is, equation (1.8) has a unique bounded solution, satisfying the initial condition

$$v(0) = c. \tag{1.9}$$

On the other hand, it is easy to see that in this case for the boundedness of any solution $v : \mathbb{R}_+ \to \mathbb{R}$ to equation (1.8) it is necessary and sufficient that this solution satisfies the inequality

$$v'(t)v(t) \le 0 \quad \text{for} \quad t \in \mathbb{R}_+.$$

(Such a solution is said to be a **Kneser solution**). It is also easy to see that if

$$\int_0^{+\infty} tf(t,x)\,dt = +\infty, \tag{1.10}$$

then any Kneser solution to equation (1.8) is vanishing at infinity.

Therefore, Kneser actually proved the following

Theorem 1.1. *If $f \in K_{loc}(\mathbb{R}_+ \times \mathbb{R})$ is a nondecreasing in the second argument function and*

$$f(t,0) \equiv 0,$$

then for any real number c problem (1.8), (1.9) *has a unique Kneser solution. And if f in addition satisfies condition* (1.10)*, then that solution is vanishing at infinity.*

It seems that the author was motivated to obtain this result by purely mathematical interest, and not by any consideration of its practical application. However, three decades after the publication of Knezer's work, the problem he posed really found an interesting application in physics. In 1927, the outstanding physicists Fermi and Thomas, in connection with the question of the distribution of electrons in a heavy atom, in their works [3], [10] considered the boundary value problem

$$u'' = t^{-\frac{1}{2}}|u|^{\frac{3}{2}}\operatorname{sgn}(u), \tag{1.11}$$

$$u(0) = 1, \quad \lim_{t\to+\infty} u(t) = 0, \tag{1.12}$$

and independently of each other proved its unique solvability. They were not familiar with Kneser's paper and did not know that their result was a rather particular case of the above given Theorem 1.1. Indeed, if

$$f(t,x) \equiv t^{-\frac{1}{2}}|x|^{\frac{3}{2}}\operatorname{sgn}(x),$$

then from Theorem 1.1 it follows the following simple

Corollary 1.1. *Equation* (1.11) *has a unique Kneser solution, satisfying the initial condition*

$$u(0) = 1, \tag{1.12$'$}$$

and this solution is vanishing at infinity.

According to this result, problem (1.11), (1.12) is not only uniquely solvable but it is also equivalent to problem (1.11), $(1.12')$ in a class of Kneser solutions.

True, the above-cited Fermi and Thomas papers do not contain new mathematical result, but they significantly determined the interest of specialists in Kneser type problems. The results obtained in this direction are contained in the review work by I. Kiguradze and B. Shekhter [8].

Note that the terms ”Kneser solution” and ”Kneser's problem”, already established in the literature today, for both second and higher order differential equations and systems were introduced by I. Kiguradze.

Investigations by I. Kiguradze and T. Chanturia (see, [1, 2, 4, 5, 6]) played a significant role in the construction of a qualitative theory of monotone solutions to nonautonomous differential equations and systems.

The results on the existence, uniqueness and asymptotic behavior of Kneser solutions to problems (1.1), (1.2) and $(1.1')$, (1.2), given in the present report, are obtained jointly with I. Kiguradze. They fundamentally differ from previously known results (see, [7, 8]) in that they include cases when the differential systems under consideration have a non-integrable singularity in the time variable at the initial point.

2. Problem (1.1), (1.2)

When discussing this problem, it is always assumed that conditions (1.3) are satisfied, $\boldsymbol{r}$ is an arbitrarily fixed positive number, and $\boldsymbol{f_1^*}$, $\boldsymbol{f_{1*}}$, $\boldsymbol{f_{2*}}$ are functions defined by the following equalities:

$$\begin{aligned} \boldsymbol{f_1^*(t,x)} &= \max\left\{|\boldsymbol{f_1(t,x_1,x_2)}| : \ \boldsymbol{0 \le x_1 \le r}, \quad \boldsymbol{-x \le x_2 \le 0}\right\} \\ &\qquad\qquad\qquad\qquad \text{for} \quad \boldsymbol{t > 0}, \quad \boldsymbol{x \in \mathbb{R}_+}, \\ \boldsymbol{f_{1*}(t,x)} &= \inf\left\{|\boldsymbol{f_1(t,x_1,x_2)}| : \ \boldsymbol{0 \le x_1 \le r}, \quad \boldsymbol{x_2 \le -x}\right\} \\ &\qquad\qquad\qquad\qquad \text{for} \quad \boldsymbol{t > 0}, \quad \boldsymbol{x \in \mathbb{R}_+}, \\ \boldsymbol{f_{2*}(t,x)} &= \inf\left\{|\boldsymbol{f_2(t,x_1,x_2)}| : \ \boldsymbol{x \le x_1 \le r}, \quad \boldsymbol{x_2 \in \mathbb{R}_-}\right\} \\ &\qquad\qquad\qquad\qquad \text{for} \quad \boldsymbol{t > 0}, \quad \boldsymbol{0 \le x \le r}. \end{aligned}$$

Theorem 2.1. *Let there exist positive numbers* a, ℓ_0, *and nonnegative functions*

$$\ell \in L_{loc}(\,]0,a]), \quad \ell_1 \in L([0,a]) \tag{2.1}$$

such that along with (1.4) *the conditions*

$$f_2(t,x_1,x_2) \le \ell(t) + \big(\ell_1(t) + \ell_0|f_1(t,x_1,x_2)|\big)(1+|x_2|) \tag{2.2}$$
$$\text{for} \quad 0<t<a, \quad 0\le x_1\le r, \quad x_2 \in \mathbb{R}_-,$$

$$\lim_{x\to+\infty} \int_0^a f_{1*}(t,x)\,dt > r, \tag{2.3}$$

$$\int_0^a \Big(f_1^*\Big(t, x\int_t^a \ell(s)\,ds\Big)\Big)dt < +\infty \quad \text{for} \quad x>0 \tag{2.4}$$

hold. Then for every $c \in [0,r]$ *problem* (1.1), (1.2) *has at least one Kneser solution.*

If system (1.1) has a Kneser solution, then the questions naturally arise: in what cases are these solutions vanishing at infinity and in what cases are they strongly Kneser solutions? The following statements answer these questions.

Theorem 2.2. *Let along with* (1.4), *for every arbitrarily small positive number* x, *one of the following three conditions hold:*

$$\int_1^{+\infty} f_{i*}(t,x)\,dt = +\infty \quad (i=1,2); \tag{2.5}$$

$$\int_1^{+\infty} f_{1*}(t,x)\,dt < +\infty, \quad \int_1^{+\infty} f_{2*}\Big(t, \int_t^{+\infty} f_{1*}(s,x)\,ds\Big)\,dt = +\infty; \tag{2.6}$$

$$\int_1^{+\infty} f_{2*}(t,x)\,dt < +\infty, \quad \int_1^{+\infty} f_{1*}\Big(t, \int_t^{+\infty} f_{2*}(s,x)\,ds\Big)\,dt = +\infty. \tag{2.7}$$

Then for every $c \in [0,r]$ *a Kneser solution to problem* (1.1), (1.2) *is vanishing at infinity.*

Proposition 2.1. *For every nontrivial Kneser solution to system* (1.1) *to be strongly Kneser solution, it is sufficient that there exist a positive constant* ε *and a non-negative function* $h \in L_{loc}(\,]0,+\infty[\,)$ *such that the functions* f_1 *and* f_2 *satisfy the inequality*

$$\sum_{i=1}^{2} |f_i(t,x_1,x_2)| \le h(t)(|x_1|+|x_2|) \quad \textit{for} \quad t>0, \ \ |x_1|+|x_2| \le \varepsilon.$$

Example 2.1. Let

$$f_1(t,x_1,x_2) \equiv -\big(|x_1|^{\lambda_1} + |x_2|^{\lambda_2}\big),$$
$$f_2(t,x_1,x_2) \equiv \ell_0 |f_1(t,x_1,x_2)|(1+|x_2|)^{\lambda},$$

where λ_1, λ_2, λ, and ℓ_0 are positive constants. Then

$$f_1^*(t,x) \equiv -\big(r^{\lambda_1} + x^{\lambda_2}\big), \quad f_{1*}(t,x) \equiv x^{\lambda_1}, \quad f_{2*}(t,x) \equiv \ell_0 x^{\lambda_1},$$

and system (1.1) has the form

$$u_1' = -\big(|u_1|^{\lambda_1} + |u_2|^{\lambda_2}\big), \quad u_2' = \ell_0\big(|u_1|^{\lambda_1} + |u_2|^{\lambda_2}\big)(1+|u_2|)^{\lambda}. \tag{2.8}$$

Hence by Theorems 2.1 and 2.2 it follows that if

$$\lambda \leq 1$$

and $c \in [0,r]$, then problem $(2.8),(1.2)$ has at least one vanishing at infinity Kneser solution.

Assume now that problem $(2.8),(1.2)$ has a Kneser solution also in the case where

$$\lambda > 1.$$

Then according to Theorem 2.2 this solution is vanishing at infinity. Thus

$$\int_0^{+\infty} \big(|u_1(t)|^{\lambda_1} + |u_2(t)|^{\lambda_2}\big)\, dt = c,$$

$$\frac{1 - \big(1 + |u_2(0)|\big)^{1-\lambda}}{\lambda - 1} = \ell_0 \int_0^{+\infty} \big(|u_1(t)|^{\lambda_1} + |u_2(t)|^{\lambda_2}\big)\, dt = \ell_0 c,$$

and

$$c < \frac{1}{\ell_0(\lambda - 1)}.$$

Therefore, no matter how small $\lambda - 1$ is, if

$$r \geq c \geq \frac{1}{\ell_0(\lambda - 1)},$$

then problem (1.1), (1.2) has no Kneser solution.

The above-given example shows that condition (2.2) in Theorem 2.1 cannot be replaced by the condition

$$f_2(t, x_1, x_2) \leq \ell(t) + \big(\ell_1(t) + \ell_0 |f_1(t, x_1, x_2)|\big)(1 + |x_2|)^{1+\varepsilon},$$

no matter how small $\varepsilon > 0$ is.

Example 2.2. Let

$$f_1(t,x_1,x_2) \equiv -\big(|x_1|^{\lambda_1} + |x_2|^{\lambda_2}\big),$$
$$f_2(t,x_1,x_2) \equiv p(t)x_1 + \ell_0|f_1(t,x_1,x_2)||x_2|,$$

where

$$\lambda_1 \geq 1, \quad \lambda_2 \geq 1, \quad \ell_0 \geq 0,$$

and $p \in L_{loc}(\,]0,+\infty[\,)$ is a nonnegative function. Then system (1.1) has the form

$$u_1' = -\big(|u_1|^{\lambda_1} + |u_2|^{\lambda_2}\big), \quad u_2' = p(t)u_1 + \ell_0\big(|u_1|^{\lambda_1} + |u_2|^{\lambda_2}\big)|u_2|. \tag{2.9}$$

By Theorem 2.2 and Proposition 2.1, if

$$\int_0^a \Big(\int_t^a p(s)\,ds\Big)^{\lambda_2} dt < +\infty, \tag{2.10}$$

where $a > 0$, then for every $c > 0$ problem (1.1), (1.2) has at least one strongly Kneser solution.

Assume now that (u_1, u_2) is a Kneser solution to problem (2.9), (1.2) for some positive number c. Then, according to Proposition 2.1, this solution is a strongly Kneser solution and, therefore, for every $a > 0$ the inequality

$$u_1(t) > \delta \quad \text{for} \quad 0 < t < a$$

holds, where
$$\delta = u_1(a) > 0.$$
On the other hand, almost everywhere on $]0,+\infty[$ we have
$$-u_1'(t) > |u_2(t)|^{\lambda_2}, \quad |u_2(t)|' = -p(t)u_1(t) + \ell_0 u_1'(t)|u_2(t)|.$$
Thus
$$|u_2(t)| = \exp\big(\ell_0(u_1(t)-\delta)\big)|u_2(a)| +$$
$$\int\limits_t^a \exp\big(\ell_0(u_1(t)-u_1(s))\big)p(s)u_1(s)\,ds > \delta\int\limits_t^a p(s)\,ds \quad \text{for } \ 0<t<a,$$
and
$$c-\delta > \int\limits_0^a |u_2(t)|^{\lambda_2}dt > \delta^{\lambda_2}\int\limits_0^a \Big(\int\limits_t^a p(s)\,ds\Big)^{\lambda_2}dt.$$
Therefore, for problem (2.9), (1.2) to have at least one Kneser solution for every $c > 0$, it is necessary and sufficient that condition (2.10) be satisfied.

The above-given example shows that condition (2.4) in Theorem 2.1 is unimprovable.

Remark 2.1. If $c \in]0, r]$ and

$$f_i(t, x_1, x_2) \equiv p_i(t)|x_{3-i}|^{\lambda_i} \operatorname{sgn}(x_{3-i}) \quad (i = 1, 2),$$

where

$$\lambda_1 > 0, \quad \lambda_2 \geq 1/\lambda_1,$$

and $p_i \in L_{loc}(]0, +\infty[)$ $(i = 1, 2)$ are nonnegative functions, then for a Kneser solution to problem (1.1), (1.2) to be vanishing at infinity, it is necessary and sufficient that one of the conditions (2.5)–(2.7) be satisfied. Therefore, the conditions of Theorem 2.2 are unimprovable.

3. Problem $(\mathbf{1.1'})$, (1.2)

Throughout this section it is assumed that conditions (1.5) are satisfied and $\boldsymbol{c}$ is an arbitrarily fixed real number.

Theorem 3.1. *Let the functions* $\boldsymbol{g_1}$ *and* $\boldsymbol{g_2}$ *along with* (1.6) *satisfy one of the following two conditions:*

$$\int\limits_{1}^{+\infty} \boldsymbol{g_1(t,x)\,dt = +\infty} \quad \textit{for} \quad \boldsymbol{x > 0;} \tag{3.1}$$

$$\int\limits_{1}^{+\infty} \boldsymbol{g_1(t,x)dt<+\infty}, \quad \int\limits_{1}^{+\infty} \boldsymbol{g_2\Big(t, \int\limits_{t}^{+\infty} g_1(s,x)ds\Big)dt=+\infty} \quad \textit{for} \quad \boldsymbol{x>0.} \tag{3.2}$$

Then for problem $(\mathbf{1.1'})$, (1.2) *to have a unique Kneser solution for an arbitrarily fixed real number* $\mathbf{c}$*, it is necessary and sufficient that the condition*

$$\int\limits_{0}^{1} \boldsymbol{g_1\Big(t, x + \int\limits_{t}^{1} g_2(s,x)\,ds\Big)\,dt < +\infty} \quad \textit{for} \quad \boldsymbol{x > 0} \tag{3.3}$$

be satisfied.

Remark 3.1. The requirement that one of the conditions (3.1) and (3.2) be fulfilled in the above-formulated theorem is essential and it cannot be replaced by the condition

$$0 < \int_t^{+\infty} g_1(s,x)\,ds < +\infty \quad \text{for} \quad t > 0, \quad x > 0. \tag{3.4}$$

Indeed, let $g_2(t,x) \equiv 0$, and $g_1 \in K_{loc}(\mathbb{R}_+ \times \mathbb{R})$ be a nondecreasing in the second argument and odd function, satisfying condition (3.4). Then conditions (1.6) and (3.3) hold but both conditions (3.1) and (3.2) are violated. On the other hand, for any positive constant c_1, satisfying the inequality

$$\int_0^{+\infty} g_1(s,c_1)\,ds < c,$$

a vector-function (u_1, u_2) with the components

$$u_1(t) \equiv c - \int_0^t g_1(s,c_1)\,ds, \quad u_2(t) \equiv -c_1$$

is a Kneser solution to problem $(1.1')$, (1.2). Therefore, this problem has an infinite set of Kneser solutions.

Remark 3.2. As we have seen, conditions (1.6), (3.1), and (3.3) guarantee the unique solvability of problem $(1.1')$, (1.2). On the other hand, these conditions do not guarantee that a solution of that problem is vanishing at infinity. Indeed, let $g_2(t,x) \equiv 0$, and the function $g_1 \in K_{loc}(\mathbb{R}_+ \times \mathbb{R})$ satisfy conditions (1.6) and (3.1). Then condition (3.3) is automatically satisfied, and a vector-function (u_1, u_2) with the components

$$u_1(t) \equiv c, \quad u_2(t) \equiv 0$$

is a Kneser solution to problem $(1.1')$, (1.2) which is not vanishing at infinity.

In the next theorem, concerning the fact that a Kneser solution to problem $(1.1')$, (1.2) is vanishing at infinity, instead of (3.1) one of the following two conditions is required:

$$\int_1^{+\infty} g_i(t,x)\,dt = +\infty \quad (i = 1,2); \tag{3.5}$$

$$\int_1^{+\infty} g_2(t,x)dt < +\infty, \quad \int_1^{+\infty} g_1\Big(t, \int_t^{+\infty} g_2(s,x)ds\Big)dt = +\infty \quad \text{for } \ x > 0. \tag{3.6}$$

Theorem 3.2. *Let the functions* g_1 *and* g_2 *along with conditions* (1.6) *and* (3.3) *satisfy one of the conditions* (3.2), (3.5), *and* (3.6). *Then problem* $(1.1')$, (1.2) *has a unique Kneser solution and this solution is vanishing at infinity.*

The particular cases of problem $(1.1')$, (1.2) are the Emden–Fowler type differential system

$$u_1' = p_1(t)|u_2|^{\lambda_1}\mathrm{sgn}\,(u_2), \quad u_2' = p_2(t)|u_1|^{\lambda_2}\mathrm{sgn}\,(u_1) \tag{3.7}$$

with the initial condition (1.2), and the differential equations

$$u'' = p(t)|u|^{\lambda}\mathrm{sgn}\,(u), \tag{3.8}$$

$$u'' = f(t, u) \tag{3.9}$$

with the initial condition

$$\lim_{t\to a} u(t) = c. \tag{3.10}$$

Here λ_1, λ_2, and λ are positive constants,

$$p_1 \in L_{loc}(\mathbb{R}_+), \quad p_2 \in L_{loc}(\,]0, +\infty[\,), \quad p \in L_{loc}(\,]0, +\infty[\,)$$

are nonnegative functions, and

$$f \in K_{loc}(\,]0, +\infty[\,\times\mathbb{R})$$

is a nondecreasing in the second argument and odd function.

Theorems 3.1 and 3.2 imply the following statements.

Corollary 3.1. *Let* $c \neq 0$. *For problem* (3.7), (1.2) *to have a unique Kneser solution, it is necessary and sufficient that the functions* p_1 *and* p_2 *satisfy the condition*

$$\int_0^1 p_1(t)\Big(\int_t^1 p_2(s)\,ds\Big)^{\lambda_1} dt < +\infty, \tag{3.11}$$

as well as one of the following two conditions:

$$\int_1^{+\infty} p_1(t)\,dt = +\infty, \tag{3.12}$$

$$\int_1^{+\infty} p_1(t)\,dt < +\infty, \quad \int_1^{+\infty} p_2(t)\Big(\int_t^{+\infty} p_1(s)\,ds\Big)^{\lambda_2} dt = +\infty. \tag{3.13}$$

Corollary 3.2. *Let* $c > 0$. *Then for problem* (3.7), (1.2) *to have a vanishing at infinity Kneser solution, it is sufficient and in the case* $\lambda_1 \lambda_2 \geq 1$ *it is also necessary that the functions* p_1 *and* p_2 *along with condition* (3.11) *satisfy either condition* (3.13) *or one of the following two conditions:*

$$\int_1^{+\infty} p_i(t)\,dt = +\infty \quad (i = 1, 2); \tag{3.14}$$

$$\int_1^{+\infty} p_2(t)\,dt < +\infty, \quad \int_1^{+\infty} p_1(t)\Big(\int_t^{+\infty} p_2(s)\,ds\Big)^{\lambda_1} dt = +\infty. \tag{3.15}$$

Corollary 3.3. *For problem* (3.8), (3.10) *to have a unique Kneser solution, it is necessary and sufficient that the function* p *satisfy the condition*

$$\int_0^1 t\,p(t)\,dt < +\infty. \tag{3.16}$$

Corollary 3.4. *For problem* (3.8), (3.10) *to have a vanishing at infinity unique Kneser solution, it is sufficient and in the case* $\lambda \geq 1$ *it is also necessary that along with condition* (3.16) *the following condition*

$$\int_1^{+\infty} t\, p(t)\, dt = +\infty \tag{3.17}$$

be satisfied.

As an example, we consider the Fermi–Thomas type differential equation

$$u'' = p_0(t) t^{-\alpha} |u|^\lambda \operatorname{sgn}(u), \tag{3.18}$$

where α and λ are positive constants, and $p_0 : \mathbb{R}_+ \to]0, +\infty[$ is a measurable bounded function, far from zero.

Corollaries 3.3 and 3.4 imply

Corollary 3.5. *For problem* (3.18), (3.10) *to have a vanishing at infinity unique Kneser solution, it is necessary and sufficient that the inequality*

$$\alpha < 2$$

be satisfied.

As we have already noted, Fermi and Thomas considered the case, where

$$p_0(t) \equiv 1, \quad \alpha = \frac{1}{2}, \quad \lambda = \frac{3}{2}.$$

The last result of our report concerns problem (3.9), (3.10) and generalizes the above Kneser Theorem 1.1 in the case, where the function f has a non-integrable singularity in the time variable at the point $t = 0$.

Corollary 3.6. *For problem* (3.9), (3.10) *to have a unique Kneser solution for every* $c > 0$*, it is necessary and sufficient that the condition*

$$\int_0^1 t\, f(t, x)\, dt < +\infty \quad \textit{for} \quad x > 0$$

be satisfied. If along with this condition the following condition

$$\int_1^{+\infty} t\, f(t, x)\, dt = +\infty$$

holds, then that Kneser solution is vanishing at infinity.

References

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Thank You for Your Attention!