Two Point Boundary Value Problems For The Fourth Order Ordinary Differential Equations At Resonance Case.

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Consider fourth order ordinary differential equation

$$u^{(4)}(t) = p(t)u(t) + f(t,\, u(t)) + h(t) \tag{1}$$

where p is a Lebseque integrable **constantsign** function and f is a Caratheodory's class function. We study this equation under the two-point boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \;\; (j = 0, 1), \tag{2}$$

at the resonance case, i.e. when linear homogeneous equation

$$w^{(4)}(t) = p(t)w(t) \tag{3}$$

under the conditions

$$w^{(j)}(a) = 0, \quad w^{(j)}(b) = 0 \;\; (j = 0, 1), \tag{4}$$

has nonzero solution. Our goal will be to establish Landesman-Laser type conditions for solvability of our problem, which in a certain sense is the Fredholm theorem's generalisation.

For problem (1), (2) we always assume that

$$p(t) \succcurlyeq 0 \qquad \text{for} \quad t \in I,$$

becaus as it is well known if $p(t) \leq 0$ then problem (3), (4) has only the trivial solution and therefore our nonlinear problem can not be at resonance.

We will consider problem (1), (2) for two cases.

a. Every nonzero solution of problem (3), (4) change's its sign on the interval I;

b. Every nonzero solution of problem (3), (4) is a constantsign function on the interval I, or in other words when

$$p \in D_+(I),$$

where the class $D_+(I)$ is defined as in previous talk:

***D e f i n i t i o n* 1.** We will say that $p \in D_+(I)$ if $p \in L(I; R_0^+)$, and problem (3), (4) has a solution u such that

$$u(t) > 0 \;\; \text{for} \quad t \in]a,\, b[\,. \tag{5}$$

For more simplicity we will consider equation (1) only under condition (2), but analogous results are true for the boundary conditions

$$u^{(j)}(a) = 0 \; (j = 0, 1, 2), \quad u(b) = 0, \tag{6}$$

where the main difference is that for problem (1), (6) we always assume that

$$p(t) \preccurlyeq 0 \qquad \text{for} \quad t \in I,$$

because in the opposite case problem (3), (6) has only the trivial solution, and we will use the class $D_-(I)$ defined in the previous talk.

The case when nonzero solutions of problem (3), (4) are constant sign functions on the interval I we consider in details, and we will give proof of the theorem schematically.

It is interesting that for this case we have also the theorem of **unique solvability** of problem (3), (4) at resonance case.

When all the solutions of problem (3), (2) change the sign on I, we only show the main theorem.

Now let's consider the mentioned results, for this we need the following definitions:

1. $f^*(t,\, r) := \sup\{|f(t,\, x)| : |x| \le r\} \in L([a,\, b];\, \mathbb{R}_+)$;
2. If w is a nonzero solution of problem (3), (2), then assume that

$$N_p := \{t \in]a,\, b[:\ w(t) = 0,\ w \not\equiv 0\}.$$

Also we need the following definition:

***D e f i n i t i o n* 2.** Let $f \in K(I \times \mathbb{R};\, \mathbb{R})$. Then we will say, that

$$f \in E(N_p),$$

if for any neighbourhood $U(N_p)$ of the set N_p and positive constant $r > 0$, there exists a constant $\lambda_1 > 0$ such, that

$$\int_{U(N_p)\setminus U_\lambda(N_p)} |f(s,x)|ds - \int_{U_\lambda(N_p)} |f(s,x)|ds \ge 0 \qquad \text{if} \quad |x| \ge r,\ \lambda \le \lambda_1, \tag{7}$$

where $U_\lambda(N_p)$ is the neighbourhood of the set N_p with the radius λ.

***T h e o r e m* 1.** Let

$$N_p \ne \emptyset \qquad \text{and} \qquad f \in E(N_p), \tag{8}$$

$r > 0$, and functions $f^+, f^- \in L(I;\, \mathbb{R}_0^+)$ be such that

$$\begin{aligned} f(t,x) &\le -f^-(t) \qquad \text{for} \quad x \le -r, \quad t \in I, \\ f^+(t) &\le f(t,x) \qquad \text{for} \quad x \ge r, \quad t \in I, \end{aligned} \tag{9_i}$$

and the condition

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b f^*(s,\rho)ds = 0 \tag{10}$$

holds.

Let, moreover w is a nonzero solution of problem (3), (4) and there exists $\varepsilon > 0$ such that

$$\begin{aligned} -\int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds + \varepsilon\gamma_r ||w||_C \le \\ \le \int_a^b h(s)w(s)ds \le \\ \le \int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds - \varepsilon\gamma_r ||w||_C, \end{aligned} \tag{11}$$

where $\gamma_r = \int_a^b f^*(s,\, r)ds$. Then problem (1), (2) is solvable.

Conditions (10), (9_i) show us that the function f is sublinear in the second argument and bounded from one side by the function f^- if $x < -r$ and by the function f^+ if $x > r$ respectively.

Condition (11) is the Landesman-Lazer type condition, which for $f \equiv 0$, i.e. when equation (1) transforms to the nonhomogeneous linear equation

$$u^{(4)}(t) = p(t)u(t) + h(t)$$

transforms to the condition

$$\int_a^b h(s)w(s)ds = 0$$

which, as we now from the Fredholms theorem, guarantee the solvability of the problem

$$u^{(4)}(t) = p(t)u(t) + h(t) \qquad u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \ \ (j = 0, 1).$$

Let us consider now the case

$$N_p = \emptyset,$$

which means that nonzero solutions of our linear homogeneous problem are constantsign functions.

***Theorem* 2.** Let

$$p \in D_+(I) \qquad \text{and} \quad p(t) > 0 \ \text{ on } \ I, \tag{12}$$

and there exists constanr $r > 0$ and functions $f^-,\ f^+,\ g \in L(I; \mathbb{R}_0^+)$ such that conditions

$$\begin{array}{ll} f^-(t) \leq f(t,x) \leq g(t)|x| & \text{for} \quad x < -r, \ \ t \in I, \\ -g(t)|x| \leq f(t,x) \leq -f^+(t) & \text{for} \quad x > r, \ \ t \in I, \end{array} \tag{13}$$

hold. Let moreover w is a nonzero solution of problem (3), (4) and there exits constant $\varepsilon > 0$ such that the Landesman-Lather's type condition

$$-\int_a^b f^-(s)|w(s)|ds + \varepsilon\gamma_r||w||_C \leq \int_a^b h(s)|w(s)|ds \leq \int_a^b f^+(s)|w(s)|ds - \varepsilon\gamma_r||w||_C, \tag{14}$$

holds, where $\gamma_r = \int_a^b f^*(s,\, r)ds$. Then problem (1), (2) is solvable.

Condition (13) show us that when $x < -r$ then the positive part of the function $f(t,x)$ is free of the condition of sublinearity and when $x > r$ then the negative part of the function $f(t,x)$ is free of the condition of sublinearity.

***Remark* 1.** If $f \not\equiv 0$, then condition (14) of Theorem 2 can be replaced by the simpler condition

$$-\int_a^b f^-(s)|w(s)|ds < \int_a^b h(s)|w(s)|ds < \int_a^b f^+(s)|w(s)|ds.$$

It should be noted that: the development of the research presented in the previous report was awakened only by the needs of proving this theorem. That's exactly why I will give the proof of this theorem, but I'll omit some details.

Proof. For an arbitrary $n \in \mathbb{N}$ we define the function p_n as

$$p_n(t) \equiv \Big(1 - \frac{1}{n}\Big)p(t).$$

From the inclusion $p \in D_+(I)$ it is clear that $p \not\equiv 0$, and therefore

$$[p_n(t)]_+ = p_n(t) \preccurlyeq p(t) \qquad \text{for} \quad t \in I, \ n \in \mathbb{N}.$$

Also from condition (13) immediately follows that

$$-g(t)|x| \le f(t,x)\,\mathrm{sgn}\,x \le 0 \quad \text{for} \quad |x| > r, \tag{15}$$

and then for an arbitrary $n \in \mathbb{N}$ by the theorem from the previous talk we get that the problem

$$u_n^{(4)}(t) = p_n(t)u_n(t) + f(t, u_n(t)) + h(t) \qquad \text{for} \quad t \in I, \tag{16}$$

$$u_n^{(i-1)}(a) = 0, \qquad u_n^{(i-1)}(b) = 0 \ (i = 1, 2), \tag{17}$$

has at list one solution u_n. Also in view the condition (15) the estimation

$$|f(t,\, x)| \le f^*(t,\, r) + g(t)|x| \qquad \text{for} \quad t \in I, \;\; x \in \mathbb{R} \tag{18}$$

is true.

Now our goal is to show that there exists such a positive constant ρ that

$$||u_n||_C \le \rho \quad \text{for all} \quad n \in \mathbb{N}.$$

Let assume the contrary, that

$$\lim_{n\to+\infty} ||u_n||_C = +\infty. \tag{19}$$

Now consider functions $v_n(t) := u_n(t)||u_n||_C^{-1}$, then it is clear that

$$||v_n||_C = 1 \qquad \text{for} \quad n \in \mathbb{N}, \tag{20}$$

and if we divide equation (16) by the norm $||u_n||_C$, by the elementary transformations we will get that v_n is a solution of the problem

$$v_n^{(4)}(t) = [p_n(t) + p_0(t, u_n(t))]v_n(t) + \frac{1}{||u_n||_C}\, p_1(t, u_n(t)) \quad \text{for all} \quad n \in \mathbb{N} \tag{21}$$

$$v_n^{(i-1)}(a) = 0, \qquad v_n^{(i-1)}(b) = 0 \ (i = 1, 2), \tag{22}$$

where

$$p_0(t,x) = \frac{1}{|x|+1}\, f(t,x)\,\mathrm{sgn}x, \qquad p_1(t,x) = h(t) + \frac{1}{|x|+1}\, f(t,x).$$

For functions p_0 and p_1 in view the inequality (18) we easily get the estimations

$$\begin{gathered} |p_0(t,\, x)| \le \sigma_0(t) := f^*(t,\, r) + g(t) \qquad \text{for} \quad t \in I, \;\; x \in \mathbb{R}, \\ |p_1(t,\, x)| \le \sigma_0(t) + |h(t)| \qquad \text{for} \quad t \in I, \;\; x \in \mathbb{R}, \\ |v_n^{(4)}(t)| \le \sigma_0(t) + p(t) + \frac{\sigma_0(t) + |h(t)|}{||u_n||_C} \qquad \text{for} \quad t \in I. \end{gathered} \tag{23}$$

Also from the boundary conditions by the Rolle's theorem we conclude that the function $v_n^{(j)}$ $(j = 0, 1, 2, 3)$ has at list one zero point $c_{j,n} \in I$. Then for points $c_{j,n} \in I$ and arbitrary points $t_1,\, t_2 \in I$ we have

$$v_n^{(j)}(c_{j,n}) = 0, \qquad |v_n^{(j)}(t_2) - v_n^{(j)}(t_1)| \le \Big| \int_{t_1}^{t_2} |v_n^{(j+1)}(s)| ds \Big| \ (j = 0, 1, 2, 3) \quad \text{for} \quad n \in \mathbb{N}.$$

Therefor from equalities $||v_n||_C = 1$ and estimates (23), we easily get that sequences

$$\{v_n^{(j)}\}_{n=1}^{+\infty} \ (j = 0, 1, 2, 3)$$

are uniformly bounded and equicontinuous. Then by Alzela-Ascoli lemma, without loss od generality it can be assumed that these sequences are uniformly convergent on I. And then there exists the function w such that

$$\lim_{n\to+\infty} v_n^{(j)}(t) = w^{(j)}(t) \ (j = 0, 1, 2, 3) \qquad \text{uniformly on} \quad I, \tag{24}$$

and then the function w admits to the boundary conditions

$$w^{(i-1)}(a) = 0, \qquad w^{(i-1)}(b) = 0 \ (i = 1, 2). \tag{25}$$

Also for the function p_0 from the inequalities (15) follows the estimation

$$0 \geq p_0(t, x) \geq -g(t)\frac{|x|}{|x|+1} \geq -g(t) \qquad \text{for} \quad t \in I, \ |x| > r.$$

from which we have

$$-g(t) \leq p_0(t, \, u_n(t)) \leq 0 \qquad \text{for} \quad t \in I_n := \{t \in I : |u_n(t)| > r\}. \tag{26}$$

Now if we define the bounded sequences $\{P_n\}$ by equalities

$$P_n(t) = \int_{(a+b)/2}^{t} p_0(s, u_n(s))ds,$$

then one of our purely technical lemma show that:

$$\lim_{n \to +\infty} mes(I \setminus I_n) = 0$$

and if $n_0 \in \mathbb{N}$ is large enough, for arbitrary numbers $t_1, t_2 \in I$ $(t_2 > t_1)$, from (26) we get the estimations

$$-\int_{t_1}^{t_2} g(s)ds \leq P_n(t_2) - P_n(t_1) \leq 0 \qquad \text{for} \quad n \geq n_0,$$

from which follows that the bounded sequence $\{P_n\}_{n > n_0}$ is uniformly equicontinuous and by the Alzela-Ascoli lemma, without loss od generality it can be assumed that these sequence is uniformly convergent on I. And Therefore there exists the function $P \in C(I; \mathbb{R})$ such that

$$\lim_{n \to +\infty} P_n(t) = P(t) \qquad \text{uniformly on} \quad I. \tag{27}$$

Then if we pass to the limit in the last inequality, we will get the estimation

$$-\int_{t_1}^{t_2} g(s)ds \leq P(t_2) - P(t_1) \leq 0,$$

from which follows that the function P is absolutely continuous and therefore there exists such a function $\widetilde{p}_k \in L(I; \mathbb{R})$, that $P(t) = \int_a^t \widetilde{p}(s)ds$ for $t \in I$. Also due to the last inequality we get the estimation

$$-g(t) \leq \widetilde{p}(t) \leq 0 \qquad \text{for} \quad t \in I. \tag{28}$$

Now if we integrate equation (21) from a to t we get

$$v_n^{(3)}(t) - v_n^{(3)}(a) = \int_a^t \Big(p_n(s) + p_0(s, u_n(s))\Big)v_n(s)ds + \frac{1}{||u_n||_C}\int_a^t p_1(s, u_n(s))ds, \tag{29}$$

and if we pass to the limit in the last equality, in view the fact that

$$\lim_{n \to +\infty} \frac{1}{||u_n||_C}\int_a^t p_1(s, u_n(s))ds = 0 \qquad \text{for} \quad t \in I \tag{30}$$

we will get that the function w is a solution of the problem

$$\begin{gathered} w^{(4)}(t) = (p(t) + \widetilde{p}(t))w(t), \\ w^{(i-1)}(a) = 0, \qquad w^{(i-1)}(b) = 0 \ (i = 1, 2). \end{gathered} \tag{31}$$

Also from equations (20) it is clear that

$$||w||_C = 1. \tag{32}$$

Now assume that $\widetilde{p} \not\equiv 0$. Then $\widetilde{p} \preccurlyeq 0$, and from the condition $p(t) > 0$ follows that

$$[p(t) + \widetilde{p}(t)]_+ \preccurlyeq p(t) \qquad \text{for} \quad t \in I.$$

But from the previous talk we know that due to condition $p \in D_+(I)$, from the least inequality follows that problem (31) has only the trivial solution, which contradicts with the equality (32). Thus our assumption is invalid and $\widetilde{p}(t) \equiv 0$, and then the function w will be the nontrivial solution of the problem

$$\begin{gathered} w^{(4)}(t) = p(t)w(t), \\ w^{(i-1)}(a) = 0, \qquad w^{(i-1)}(b) = 0 \ (i = 1, 2). \end{gathered} \tag{33}$$

Now if we multiply the equations (16) და (3) respectively by w and $-u_n$, and integrate their sum from a to b, we will get the equality

$$\int_a^b [u_n^{(4)}(s)w(s) - w^{(4)}(s)u_n(s)]ds = \int_a^b [h(s) + f(s, u_n(s))]w(s)ds - \frac{1}{n}\int_a^b p(s)u_n(s)w(s)ds.$$

On the other hand by the integration by parts due to boundary conditions we have that

$$\int_a^b [u_n^{(4)}(s)w(s) - w^{(4)}(s)u_n(s)]ds = 0,$$

and therefore from the previous equality we get

$$\int_a^b [h(s) + f(s, u_n(s))]w(s)ds = \frac{1}{n||u_n||_C}\int_a^b |p(s)|v_n(s)w(s)ds \qquad \text{for} \quad n > n_0.$$

But due to condition $\lim_{n\to+\infty} v_n(t)w(t) = w^2(t)$ there exists $n_1 > n_0$ such that

$$\int_a^b [h(s) + f(s, u_n(s))]w(s)ds > 0 \qquad \text{for} \quad n > n_1. \tag{34}$$

On the other hand, on the basis of our Landesman-lazer's condition

$$-\int_a^b (f^-(s)[w(s)]_- + f^+(s)[w(s)]_+)ds < \int_a^b h(s)w(s)ds < \int_a^b (f^+(s)[w(s)]_- + f^-(s)[w(s)]_+)ds,$$

we can show that for an arbitrary sequence of such functions u_n that

$$\lim_{n\to+\infty} ||u_n|| = +\infty \quad \text{and} \quad \lim_{n\to+\infty} \frac{u_n(t)}{||u_n||} = w(t)$$

the following inequality is true

$$\int_a^b [h(s) + f(s, u_n(s))]w(s)ds \leq 0 \qquad \text{for} \quad n > n_2,$$

if the integer n_2 is large enough.

but last two inequalities contradicts and therefore our assumption that $\lim\limits_{n\to+\infty} ||u_n|| = +\infty$ is invalid and there exists such a positive number $\rho > 0$ that

$$||u_n|| \leq \rho \quad \text{for} \quad n \in N.$$

Due to the last inequality from equation (16), we get the estimation

$$|u_n^{(4)}(t)| = p(t)\rho + f^*(t, \rho) + |h(t)| \qquad \text{for} \quad t \in I.$$

Then in view of our boundary conditions sequences $\{u_n^{(j)}\}_{n=1}^{+\infty}$ $(j = 0, 1, 2, 3)$ will be uniformly bounded and equicontinuous, and then by the Alzela-Ascoli lemma, without loss od generality it can be assumed that these sequences are uniformly convergent on I to such a function u_0 that

$$\lim_{n\to+\infty} u_n^{(j)}(t) = u_0^{(j)}(t) \ (j = 0, 1, 2, 3) \qquad \text{uniformly on} \quad I. \tag{35}$$

Now if we pass to the limit in equation (16) after integration from a to t we get that u_0 is a solution of our nonlinear problem □

For proved theorem we have the following corollary:

***Corollary* 1.** Let conditions

$$p \in D_+(I) \qquad \text{and} \quad p(t) > 0 \;\text{ on }\; I,$$

hold and there exists $r > 0$ such that

$$-g(t)|x| \leq f(t,x)\,\mathrm{sgn}\, x \leq 0 \qquad \text{თუ} \quad |x| \geq r, \;\; t \in I. \tag{36}$$

Let moreover, there exist sets of positive measure I^+, $I^- \subset I$ such that

$$\begin{aligned} &\lim_{x \to +\infty} f(t,\, x) = -\infty \qquad \text{uniformly on} \quad I^-, \\ &\lim_{x \to -\infty} f(t,\, x) = +\infty \qquad \text{uniformly on} \quad I^+. \end{aligned} \tag{37}$$

Then for an arbitrary $h \in L(I;\, \mathbb{R})$ problem (1), (2) is solvable.

and at the end of my talk I wont to show following existence and uniqueness theorem:

***Theorem* 3.** Let $f(t,\, 0) \equiv 0$, conditions

$$p \in D_+(I) \qquad \text{and} \quad p(t) > 0 \;\text{ on }\; I,$$

hold and there exists continuous function $\eta : \mathbb{R}^2 \to \mathbb{R}^+$, and function $g, \ell \in L(I;\, \mathbb{R}_0^+)$ such that $\ell \not\equiv 0$ and

$$-g(t)|x_1 - x_2| \leq [f(t,x_1) - f(t,x_2)]\,\mathrm{sgn}(x_1 - x_2) \leq -\ell(t)\eta(x_1,x_2)|x_1 - x_2| \qquad \text{თუ} \quad t \in I,\; x_1, x_2 \in \mathbb{R},$$

where

$$\lim_{|\rho| \to +\infty} |\rho|\eta(\rho,\, 0) = +\infty.$$

Then for an arbitrary $h \in L(I;\, \mathbb{R})$ problem (1), (2) is uniquely solvable.