Disconjugacy Of The Fourth Order Ordinary Differential Equations And Boundary Value Problems

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Consider on the interval $I := [a,\, b]$ the fourth order ordinary differential equations

$$u^{(4)}(t) = p(t)u(t) + q(t), \tag{1}$$

and

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)), \tag{2}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \;\; (j = 0, 1), \tag{3_1}$$

$$u^{(j)}(a) = 0 \; (j = 0, 1, 2), \quad u(b) = 0, \tag{3_2}$$

where $p \in L(I;\, R)$, $f \in K(I \times R; R)$.

My talk contains results from the following paper:

M. Manjikashvili, S. Mukhigulashvili, two-point Boundary value Problems For 4th Order Ordinary Differential Equations. Miskolc Mathematical Notes, Vol. 25, No. 1, 2024, pp. 399–409,

First of all, I will talk about some results from the works:

1. M. Manjikashvili, S. Mukhigulashvili, Necessary And Sufficient Conditions Of Disconjugacy For The Fourth Order Linear Ordinary Differential Equations. Bull. math. Soc. Sci. Math. Romanie 64(112) No.4, 2021, pp 341-353,

2. E. Bravyi, S. Mukhigulashvili, On Solvability of Two-Point Boundary Value Problems with Separating Boundary Conditions for Linear Ordinary Differential Equations and Totally Positive Kernels, International Workshop QUALITDE – 2020, December 19 – 21, 2020, Tbilisi, Georgia.

Linear Problem

Definition 1. Equation

$$u^{(4)}(t) = p(t)u(t) \quad \text{for} \quad t \in I \tag{4}$$

is said to be disconjugate (non-oscillatory) on I, if every nontrivial solution u has less then four zeros on I, the multiple zeros being counted according to their multiplicity.

Definition 2. A continuous function $G : [a, b] \times [a, b] \to R$ is called a **totally positive kernel** if all determinants

$$\begin{vmatrix} G(t_1, t_1) & G(t_1, t_2) & \dots & G(t_1, t_k) \\ G(t_2, t_1) & G(t_2, t_2) & \dots & G(t_2, t_k) \\ \dots & \dots & \dots & \dots \\ G(t_k, t_1) & G(t_k, t_2) & \dots & G(t_k, t_k) \end{vmatrix}$$

are positive for all ordered sets of points $a < t_1 < \dots < t_k < b$ for all $k \in N$.

For us the following main property of totally positive kernels is important:

Proposition 1. **(Karlin – Gantmacher – Krein)**

Let $G : [a, b] \times [a, b] \to R$ be a totally positive kernel, $r \in L(I, R_0^+)$, and the operator $T : C(I, R) \to C(I, R)$ is defined by the equality

$$T(x)(t) = \int_a^b G(t, s)r(s)x(s)ds. \tag{5}$$

Then the spectrum of the operators T is a subset of the set $[0, +\infty[$.

Therefore, all characteristic values λ of the equation

$$x(t) = \lambda \int_a^b G(t,s)r(s)x(s)ds \tag{6}$$

are positive and if $\lambda < 0$, then the last equation has only the trivial solution.

Now if we consider equations

$$u^{(4)}(t) = [p(t)]_+ u(t), \tag{7}$$

$$u^{(4)}(t) = -[p(t)]_- u(t), \tag{8}$$

where $[p]_-$ and $[p]_+$ are respectively negative and positive parts of the coefficient p, and rewrite equation (4) in a form

$$u^{(4)}(t) = [p(t)]_+ u(t) - [p(t)]_- u(t), \tag{9}$$

then due to the representations

$$u(t) = -\int_a^b G_+(t,s)[p(s)]_- u(s)ds \quad \text{and} \quad u(t) = \int_a^b G_-(t,s)[p(s)]_+ u(s)ds$$

where G_+ is Green's function of problem (7), (3_1) or (7), (3_2) and G_- is Green's function of problem (8), (3_1) or (8), (3_2) , the last proposition can be translated as a following:

P r o p o s i t i o n 2.

a. If $G_+ : [a,b] \times [a,b] \to R$ is Green's function of problem (7), (3_1) ((7), (3_2)), and G_+ is a totally positive kernel, then problem (9), (3_1) ((9), (3_2)) is uniquely solvable for an arbitrary $[p]_-$.

b. If $G_- : [a,b] \times [a,b] \to R$ is Green's function of problem (8), (3_1) ((8), (3_2)), and $-G_-$ is a totally positive kernel, then problem (9), (3_1) ((9), (3_2)) is uniquely solvable for an arbitrary $[p]_+$.

Proposition 3. **(Gantmacher-Krein)**

Let $i \in \{1, 2\}, p \in L(I, R)$ be such that equation $u^{(4)} = pu$ is disconjugate on I, and G is Green's function of problem $u^{(4)} = pu, (3_i)$. Then

$$(-1)^{i-1}G$$

is the **totally positive kernel**.

Last two propositions result in the main theorem of the work:

2. E. Bravyi, S. Mukhigulashvili, On Solvability of Two-Point Boundary Value Problems with Separating Boundary Conditions for Linear Ordinary Differential Equations and Totally Positive Kernels, International Workshop QUALITDE – 2020, December 19 – 21, 2020, Tbilisi, Georgia.

Theorem 1.

a. Let the equation

$$u^{(4)}(t) = [p(t)]_+ u(t)$$

be disconjugate on I. Then problem (1), (3_1) is uniquely solvable for arbitrary $[p]_-$ and q.

b. Let the equation

$$u^{(4)}(t) = -[p(t)]_- u(t)$$

be disconjugate on I. Then problem (1), (3_2) is uniquely solvable for arbitrary $[p]_+$ and q.

For the formulation of the results we need the following two definitions of classes $D_+(I)$ and $D_-(I)$.

D e f i n i t i o n 3. We will say that $p \in D_+(I)$ if $p \in L(I; R_0^+)$, and problem (4), (3_1) has a solution u such that

$$u(t) > 0 \quad \text{for} \quad t \in]a, b[\,. \tag{10}$$

D e f i n i t i o n 4. We will say that $p \in D_-(I)$ if $p \in L(I; R_0^-)$, and problem (4), (3_2) has a solution u such that inequality (10) holds.

T h e o r e m 2. Let $p \in L(I;\, R_0^+)$. Then for the discojugacy of the equation

$$u^{(4)}(t) = p(t)u(t) \tag{4}$$

on I it is **necessary and sufficient** the existence of $p^* \in D_+(I)$, such that

$$p(t) \precnsim p^*(t) \quad \text{for} \quad t \in I. \tag{12}$$

The inequality $x \precnsim y$ means that $x \le y$ and $x \not\equiv y$.

Proof. Here we need two definitions of points $\eta(x, p)$ and $\tau(x, p)$.

D e f i n i t i o n 5. Let $t_0 \in R_0^+$, and $F(t_0,\, p_1)$ be the set of such $t_1 > t_0$ for which some solutions of equation

$$u^{(4)}(t) = p_1(t)u(t) \quad \text{for} \quad t \in R_0^+, \tag{13}$$

in the interval $[t_0,\, t_1]$ have at least 4 zeroes (according to their multiplicities). Then we will say that for equation (13), $\eta(t_0,\, p_1) = \inf F(t_0,\, p_1)$ is the first conjugate point to t_0.

D e f i n i t i o n 6. Let $t_0 \in R_0^+$, and $E(t_0,\, p_1)$ be the set of such $t_1 > t_0$ for which there exists a solution u of equation (13) such that

$$u(t_0) = u(t_1) = 0, \;\; u(t) > 0 \quad \text{for} \quad t \in]t_0,\, t_1[\,.$$

Then $\tau(t_0, p_1) = \sup E(t_0,\, p_1)$.

For an arbitrary function $x : [a,\ b] \to R$, we introduce the functions $x_+ : R_0^+ \to R$ by the equality

$$x_+(t) = \begin{cases} x(t) & \text{for } t \in I \\ 1 & \text{for } t \in R_0^+ \setminus I \end{cases}. \tag{15}$$

If $p \equiv 0$, then the validity of our theorem is trivial, therefore assume that $p \not\equiv 0$.

From our condition $p(t) \preccurlyeq p^*(t)$, by the following two Lemmas from the monograph:

I. Kiguradze, T. Chanturia, *Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations*, Kluwer Academic, Dordrecht (1993).

***L e m m a* 1.** Assume that $p_1 \geq 0$. Then for an arbitrary $t_0 \in R_0^+$ the equality

$$\tau(t_0,\ p_1) = \eta(t_0,\ p_1)$$

holds.

***L e m m a* 2.** Let $p_1(t) \geq p_2(t) \geq 0$ for $t \in R_0^+$. Then for an arbitrary $t_0 \in R_0^+$ the inequality

$$\tau(t_0, p_1) \leq \tau(t_0, p_2)$$

holds.

we obtain

$$\eta(a,\ p_+) = \tau(a,\ p_+) \geq \tau(a,\ p_+^*) = \eta(a,\ p_+^*),$$

where due to the inclusion $p^* \in D_+(I)$ by following lemma:

L e m m a 3. (M.Manjikashvili, S.Mukhigulashvili) The following assertions are equivalent:
$A.\ p \in D_+(I)$;
$B.\ \eta(a,\, p_+) = b.$

we have $\eta(a,\, p_+^*) = b$, and therefore

$$\eta(a,\, p_+) \geq b. \tag{16}$$

But the condition $p^* \in D_+(I)$ implies, that the problem

$$u^{(4)}(t) = p_+^*(t)u(t) \quad \text{for} \quad t \in I,\ \ u^{(i)}(a) = 0,\ \ u^{(i)}(b) = 0\,(i = 0, 1), \tag{17}$$

has a solution u positive in $]a,\, b[$.

Now **assume** that $\eta(a,\, p_+) = b$, then from Lemma

L e m m a 4. Let $p_1 \geq 0$. Then there exist a solution u of equation (13) positive on $]a,\, \eta(a,\, p_1)[$ such, that

$$u^{(i-1)}(a) = 0 \qquad u^{(i-1)}(\eta(a,\, p_1)) = 0 \ \ (i = \overline{1, 2}). \tag{18}$$

follows that the problem

$$v^{(4)}(t) = p_+(t)v(t) \quad \text{for} \quad t \in I,\ \ v^{(i)}(a) = 0,\ \ v^{(i)}(b) = 0\,(i = 0, 1), \tag{19}$$

has a solution v positive in $]a,\, b[$. Now if we multiply equations (17) and (19) respectively by v and $-u$, and integrate their sum from a to b, in view of boundary conditions (17) and (19), by integration by parts we obtain equality

$$\int_a^b (p^*(s) - p(s))u(s)v(s)ds = \int_a^b (u^{(4)}(s)v(s) - u(s)v^{(4)}(s))ds = 0,$$

which contradicts with our conditions:

$$p(t) \precneqq p^*(t), \quad \text{and} \quad u, v \ \text{ are positive in } \ ]a,\, b[\ .$$

Thus our assumption is invalid and due to (16) we have

$$\eta(a,\, p_+) > b. \tag{20}$$

Now **assume that equation** (4) **is oscillatory on** I, i.e., it has a solution u with at least four zeroes in $[a, b]$. Therefore if $t_0 \in [a, b[$ is the first zero of u, it is clear that $\eta(t_0, p_+) \in]t_0, b]$, and then due to (20) we get

$$\eta(t_0, p_+) < \eta(a, p_+),$$

and therefore $t_0 > a$. On the other hand due to Lemma from the paper

G. Johnson, The k-th conjugate point function for an even order linear differential equation, *Proc. Amer. Math. Soc.*, 42, 563-568 (1974).

***L e m m a* 5.** Let $p_1 \geq 0$, and $t_1 > t_2 > 0$. Then

$$\eta(t_1, p_1) > \eta(t_2, p_1).$$

in view the fact that $t_0 > a$, it follows the inequality

$$\eta(t_0, p_+) > \eta(a, p_+),$$

which is the contradiction with the previous inequality. Therefore our assumption is invalid and equation (4) is disconjugate on I. □

Let $\lambda_1 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \;\; u^{(j)}(0) = 0, \;\; u^{(j)}(1) = 0 \; (j = 0, 1), \tag{21}$$

then $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$, and it is well known that approximately

$$\lambda_1 \approx 4.73004.$$

Therefore from Theorem 2 we obtain effective and unimprovable condition of disconjugacy:

Corollary 1. Equation (4) is disconjugate on I if

$$0 \leq p(t) \preccurlyeq \frac{\lambda_1^4}{(b-a)^4} \quad \text{for} \quad t \in I, \tag{22}$$

and is oscillatory on I if

$$p(t) \geq \frac{\lambda_1^4}{(b-a)^4} \quad \text{for} \quad t \in I. \tag{23}$$

Even if both conditions (22) and (23) are violated, the question on the disconjugacy of equation $u^{(4)} = pu$ can be answered by the following theorem:

Theorem 3. Let $p \in L(I; R_0^+)$, and there exists $M \in R_0^+$ such that

$$M\frac{b-a}{2} + \int_a^b [p(s) - M]_+ ds \leq \frac{192}{(b-a)^3}. \tag{24}$$

Then equation (4) is disconjugate on I.

We have the example of such a coefficient p, that for $M \in]0, \operatorname{ess\,sup} p[$ condition (24) holds but it is violated if $M = \operatorname{ess\,sup} p$ and $M = 0$.

Theorem 4. Let $p \in L(I;\, R_0^-)$. Then for dsconjugacy of equation (4) on I it is **necessary and sufficient** the existence of $p_* \in D_-(I)$, such that

$$p_*(t) \preccurlyeq p(t) \quad \text{for} \quad t \in I. \tag{25}$$

Let $\lambda_2 > 0$ be the first eigenvalue of the problem

$$u^{(4)}(t) = \lambda^4 u(t), \;\; u^{(j)}(0) = 0 \; (j = 0, 1, 2), \;\; u(1) = 0, \tag{26}$$

then $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$, and it is well known that approximately

$$\lambda_2 \approx 5.553.$$

Therefore from Theorem 4 we obtain effective and unimprovable condition of disconjugacy:

Corollary 2. Equation (4) is disconjugate on I if

$$-\frac{\lambda_2^4}{(b-a)^4} \preccurlyeq p(t) \le 0 \quad \text{for} \quad t \in I, \tag{27}$$

and is oscillatory on I if

$$p(t) \le -\frac{\lambda_2^4}{(b-a)^4} \quad \text{for} \quad t \in I. \tag{28}$$

Even if both conditions (27) and (28) are violated, the question on the disconjugacy of equation (4) can be answered by the following theorem:

Theorem 5. Let $p \in L(I;\, R_0^-)$, and there exists $M \in R_0^+$ such that

$$M \frac{495}{1024}(b-a) + \int_a^b [p(s) + M]_- ds \le \frac{110}{(b-a)^3}. \tag{29}$$

Then equation (4) is disconjugate on I.

The theorems 1, 2 and 4, result in the following theorem of the solvability of problems (1), (3_i) $(i = 1, 2)$ from our last paper:

***T h e o r e m* 6.** Let $i \in \{1, 2\}$ and the function $p_0 \in L(I; R)$ be such that the equation

$$u^{(4)}(t) = [p_0(t)]_+ u(t) \qquad \text{if} \quad i = 1,$$

$$u^{(4)}(t) = -[p_0(t)]_- u(t) \qquad \text{if} \quad i = 2,$$

is diconjugate on I. Then if the inequality

$$(-1)^{i-1}[p(t) - p_0(t)] \leq 0 \qquad \text{for} \quad t \in I \tag{30}$$

holds, problem (1), (3_i) is uniquely solvable.

Proof. Let $i = 1$, then from Theorem 2 in view of disconjugacy of the equation $u^{(4)} = [p_0]_+ u$, it follows the exitance of $p^* \in D^+(I)$ such that

$$[p_0]_+ \preccurlyeq p^*.$$

On the other hand from condition (30) we have

$$[p]_+ \leq [p_0]_+,$$

and therefore from the last two inequalities we get the inequality

$$[p]_+ \preccurlyeq p^*.$$

Then due to last inequality, Theorem 2 guarantees the disconjugacy of the equation $u^{(4)} = [p]_+ u$, and therefore the solvability of the problem (1), (3_1) follows from the Theorem 1.

For $i = 2$, the proof is analogous and follows from Theorem 4. □

From the last theorem with $p_0 = [p]_+$ by Theorem 2 follows:

Corollary 3. Let there exist $p^* \in D_+(I)$ such that the inequality

$$[p(t)]_+ \preccurlyeq p^*(t) \quad \text{for} \quad t \in I \tag{31$_1$}$$

holds. Then problem (1), (3_1) is uniquely solvable.

Analogously, from the last theorem with $p_0 = -[p]_-$ by Theorem 4 follows:

Corollary 4. Let there exists $p_* \in D_-(I)$ such that the inequality

$$-[p(t)]_- \succcurlyeq p_*(t) \quad \text{for} \quad t \in I \tag{31$_2$}$$

holds. Then problem (1), (3_2) is uniquely solvable.

Remark 2. Condition (31_1) $((31_2))$ in Corollary 3 (4) is optimal in the sense that the inequality $\preccurlyeq$ $(\succcurlyeq)$ can not be replaced by the inequality $\leq$ $(\geq)$.

Now if we take into account the fact that $\frac{\lambda_1^4}{(b-a)^4} \in D_+(I)$ and $-\frac{\lambda_2^4}{(b-a)^4} \in D_-(I)$ (where $\lambda_1^4 \approx 500$ and $\lambda_2^4 \approx 949$,) then from the last two corollaries follows that the condition

$$p(t) \leq \frac{500}{(b-a)^4} \quad \left([p(t)]_- \leq \frac{949}{(b-a)^4}\right), \tag{32}$$

guarantees the solvability of problem (1), (3_1) $\big((1), (3_2)\big)$.

Nonlinear Problem

Now we consider the nonlinear fourth order ordinary differential equation

$$u^{(4)}(t) = p(t)u(t) + f(t, u(t)), \tag{34}$$

under the boundary conditions

$$u^{(j)}(a) = 0, \quad u^{(j)}(b) = 0 \;\; (j = 0, 1), \tag{35_1}$$

$$u^{(j)}(a) = 0 \; (j = 0, 1, 2), \quad u(b) = 0. \tag{35_2}$$

Theorem 7. Let $i \in \{1, 2\}$ and there exist $r \in R^+$ and $g \in L(I; R_0^+)$ such that a. e. on I the inequality

$$-g(t)|x| \leq (-1)^{i-1} f(t, x) \operatorname{sgn} x \leq \delta(t, |x|) \quad \text{for} \quad |x| > r \tag{36_i}$$

holds, where the function $\delta \in K(I \times R_0^+; R_0^+)$ is nondecreasing in the second argument and

$$\lim_{\rho \to +\infty} \frac{1}{\rho} \int_a^b \delta(s, \rho) ds = 0. \tag{37}$$

Then if the equation

$$u^{(4)}(t) = [p(t)]_+ u(t) \qquad \text{if} \quad i = 1,$$

$$u^{(4)}(t) = -[p(t)]_- u(t) \qquad \text{if} \quad i = 2,$$

is disconjugate, problem (34), (35_i) has at least one solution.

As we said the inequality $p(t) \leq 500/(b-a)^4$ guarantees the disconjugacy of the equation $u^{(4)}(t) = [p(t)]_+ u(t)$ if $i = 1$, and therefore from the last theorem we have:

Corollary 7. Let there exist $r \in R^+$ and $g \in L(I; R_0^+)$ such that a. e. on I the inequality

$$-g(t)|x| \leq f(t,x)\,\mathrm{sgn}\,x \leq \delta(t, |x|) \quad \text{for} \quad |x| > r \tag{39}$$

holds, where the function δ admits to the conditions of the Theorem 7. Then if inequality

$$p(t) \leq \frac{500}{(b-a)^4}, \tag{40}$$

holds, problem (34), (35_1) has at least one solution.

Now let compare this last corollary with Ivane Kiguradze's following theorem:

Theorem 8. (I. Kiguradze) Let the function $h \in L(I; R_0^+)$ be such that a. e. on I the inequality

$$f(t,x)\,\mathrm{sgn}\,x \leq h(t) \quad \text{for} \quad x \in R \tag{41}$$

holds, and

$$p(t) \leq \frac{\pi^4}{(b-a)^4} \approx \frac{97}{(b-a)^4}. \tag{42}$$

Then problem (34), (35_1) has at least one solution.

The following theorems of the uniqueness of the solution for our nonlinear problem which can be proved on the basis of comparison theorems 2 and 4.

***T h e o r e m* 9.** Let there exists $p^* \in D_+(I)$ such that a. e. on I the inequality

$$[f(t,x_1) - f(t,x_2)]\,\mathrm{sgn}(x_1 - x_2) < [p^*(t) - p(t)]|x_1 - x_2| \tag{43_1}$$

hold for $x_1, x_2 \in R$, $x_1 \neq x_2$. Then problem (34), (35_1) has at most one solution.

***T h e o r e m* 10.** Let there exists $p_* \in D_-(I)$ such that a. e. on I the inequality

$$[f(t,x_1) - f(t,x_2)]\,\mathrm{sgn}(x_1 - x_2) > [p_*(t) - p(t)]|x_1 - x_2| \tag{43_2}$$

hold for $x_1, x_2 \in R$, $x_1 \neq x_2$. Then problem (34), (35_2) has at most one solution.

Thank you for your attention