The Inverse Problem for Periodic Travelling Waves of the Linear 1D Shallow-Water Equations

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Introduction

The motion of small amplitude waves of a water layer with variable depth along the x-axis is described by the equations of the shallow water theory

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[h(x)u\right] = 0, \qquad \frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0, \tag{1}$$

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The shallow water equations conform a system of coupled PDEs of first order that can be easily decoupled into a single wave equation for the water flow velocity

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2}[h(x)u] = 0, \tag{2}$$

or for the surface elevation

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[h(x)\frac{\partial \eta}{\partial x}\right] = 0. \tag{3}$$

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$$q(x)\exp\Big(i\left[\omega t - \Psi(x)\right]\Big),$$

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The inverse problem:

Given a prescribed amplitude $q(x)$, can we determine a suitable bottom profile $h(x)$ allowing the equation to admit a travelling wave with amplitude $q(x)$?

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From (5), we deduce that $\left[(hq)^2\Psi'\right]' = 0$, and $(hq)^2\Psi'$ is a conserved quantity.

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Now, we insert (6) into (4) and arrive to a single second order ODE

$$(hq)'' + \omega^2 q = \frac{\alpha^2}{(hq)^3}. \tag{7}$$

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Theorem 1

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There exists a solution $h \in C_T^+$ of (7) for any $\alpha \neq 0$, $\omega \neq 0$.

Proof. By introducing the change of variables $y = hq$ into (7), we get the equation

$$y'' + \omega^2 q = \frac{\alpha^2}{y^3}.$$

Now, the result is a direct consequence of Theorem 3.12 in

[LS] A.C. Lazer, S. Solimini, *On periodic solutions of nonlinear differential equations with singularities*, Proc. American Math. Society 99, No. 1, 1987.

Surface Elevation Eq.

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Using this information in (8), we arrive at the equation

$$(hq')' + \omega^2 q = \frac{\alpha^2}{hq^3}. \tag{10}$$

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We assume that $q(x)$ is a T-periodic and positive function of class C^2 with a finite number of critical points in $[0, T]$, all of them non-degenerate, that is, if $q'(x) = 0$ then $q''(x) \neq 0$. Under this assumption, we can divide the interval $[0, T]$ into subintervals $[a, b]$ such that $q'(x)$ is of a constant sign on (a, b) and $q'(a) = q'(b) = 0$. Then, the substitution

$$u(x) = \frac{(h(x)q'(x))^2}{2\omega^4} \qquad \text{for } x \in (a, b) \tag{11}$$

transforms

$$(hq')' + \omega^2 q = \frac{\alpha^2}{hq^3} \tag{10}$$

into the equation

$$u'(x) = \frac{\lambda^2 q'(x)}{q^3(x)} - q(x)\,\mathrm{sgn}(q')\sqrt{2u(x)} \qquad \text{for } x \in (a, b), \tag{12}$$

where $\lambda = \alpha/\omega^2$.

We consider an interval $[a, b]$ such that $q \in C^2\Big([a, b]; \mathbb{R}\Big)$ satisfies

$$q(x) > 0 \quad \text{for } x \in [a, b], \quad q'(a) = 0, \quad q'(b) = 0, \quad q'(x) > 0 \quad \text{for } x \in (a, b). \tag{13}$$

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Obviously, non-negative solutions of (15) and (16) are the same.

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A solution to (16) is understood in the classical sense, that is, a function $u \in C^1\Big([a,b];\mathbb{R}\Big)$ satisfying (16) for every $x \in [a,b]$. We will investigate the properties of a solution to (16) subject to the initial condition

$$u(a) = 0. \tag{17}$$

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- When $u(b) = 0$?
- Do there exist one-sided limits

$$\ell_a \stackrel{def}{=} \lim_{x \to a+} \frac{\sqrt{2u(x)}}{q'(x)}, \qquad \ell_b \stackrel{def}{=} \lim_{x \to b-} \frac{\sqrt{2u(x)}}{q'(x)} \quad ?$$

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- What the values ℓ_a and ℓ_b are equal to?

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Theorem 2

There exists a threshold $\lambda_0 > 0$ such that

(i) if $0 < |\lambda| < \lambda_0$, the unique solution u of (16), (17) verifies $u(b) = 0$. Moreover, ℓ_a and ℓ_b are respectively the unique positive root of

$$y^2 + \frac{q(a)}{q''(a)} y - \frac{\lambda^2}{q^3(a) q''(a)} = 0. \tag{18}$$

and the smaller root of

$$y^2 - \frac{q(b)}{|q''(b)|} y + \frac{\lambda^2}{q^3(b)|q''(b)|} = 0. \tag{19}$$

(ii) if $|\lambda| = \lambda_0$, the unique solution u of (16), (17) verifies $u(b) = 0$. Moreover, ℓ_a and ℓ_b are respectively the unique positive root of (18) and a root of (19).

(iii) if $|\lambda| > \lambda_0$, the unique solution u of (16), (17) verifies $u(b) > 0$.

Consider now an interval $[\tilde{a}, \tilde{b}]$ such that $q \in C^2\left([\tilde{a}, \tilde{b}]; \mathbb{R}\right)$ satisfies

$$q(x) > 0 \quad \text{for } x \in [\tilde{a}, \tilde{b}], \quad q'(\tilde{a}) = 0, \quad q'(\tilde{b}) = 0, \quad q'(x) < 0 \quad \text{for } x \in (\tilde{a}, \tilde{b}).$$

and

$$q''(\tilde{a}) < 0, \qquad q''(\tilde{b}) > 0.$$

In this interval, Eq. (12) reads

$$u'(x) = \frac{\lambda^2 q'(x)}{q^3(x)} + q(x)\sqrt{2u(x)} \qquad \text{for } x \in (\tilde{a}, \tilde{b}). \tag{20}$$

Note that the function $\tilde{q}(x) = q(-x)$ verifies (13) and (14) in the interval $[a, b] = [-\tilde{b}, -\tilde{a}]$. Moreover, $v(x) = u(-x)$ satisfies the equation

$$v'(x) = \frac{\lambda^2 \tilde{q}'(x)}{\tilde{q}^3(x)} - \tilde{q}(x)\sqrt{2v(x)} \qquad \text{for } x \in (a, b), \tag{21}$$

which is just (15). In conclusion, the case of an interval where q is decreasing can be reduced to the case studied above.

$$(hq')' + \omega^2 q = \frac{\alpha^2}{hq^3} \tag{10}$$

Main Result

Let us assume that q is a T-periodic and positive function of class C^2 with a finite number of critical points in $[0, T]$, all of them non-degenerate, that is, if $q'(x) = 0$ then $q''(x) \neq 0$. Then, there exists a threshold $\lambda_0 > 0$ such that

(i) there exists a positive T-periodic solution h of (10) provided $0 < \left|\frac{\alpha}{\omega^2}\right| < \lambda_0$,

(ii) no positive T-periodic solution of (10) exists provided $\left|\frac{\alpha}{\omega^2}\right| > \lambda_0$.

Moreover,

$$\frac{q_*^5}{4|q_0|} < \lambda_0^2 \leq \min\left\{ \frac{q^5(b)}{4|q''(b)|} \ : \ q'(b) = 0, q''(b) < 0 \right\}, \tag{22}$$

where

$$q_* \overset{def}{=} \min\{q(x) : x \in [0, T]\}, \quad q_0 \overset{def}{=} \min\{q''(x) : x \in [0, T]\}. \tag{23}$$

Estimation of the Threshold λ_0

Theorem 3

Let there exist positive constants λ_1 and λ_2 such that $\lambda_1 \leq \lambda_2$, and let $v, w \in AC\big([a,b];\mathbb{R}\big)$ satisfy

$$v'(x) \geq \frac{\lambda_1^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|v(x)|}\,\mathrm{sgn}\, v(x) \qquad \text{for a. e. } x \in [a,b],$$

$$w'(x) \leq \frac{\lambda_2^2 q'(x)}{q^3(x)} - q(x)\sqrt{2|w(x)|}\,\mathrm{sgn}\, w(x) \qquad \text{for a. e. } x \in [a,b],$$

$$v(a) \geq 0 \geq w(a), \qquad v(b) = 0 = w(b)$$

$$\liminf_{x \to b-} \frac{\sqrt{2|w(x)|}\,\mathrm{sgn}\, w(x)}{q'(x)} > y_1(\lambda_2),$$

where $y_1(\lambda_2)$ is the smaller root of (19) with $\lambda = \lambda_2$. Then, the threshold λ_0 admits the estimate

$$\lambda_1 \leq \lambda_0 \leq \lambda_2. \tag{24}$$

Estimation of the Threshold λ_0

Corollary

Let there exist positive constants λ_1 and λ_2 such that $\lambda_1 \le \lambda_2$, let $q \in C^2\big([a,b];\mathbb{R}\big)$, and let $\ell_1, \ell_2 \in C^1\big([a,b];\mathbb{R}\big)$ satisfy

$$\ell_i(x) > 0 \qquad \text{for } x \in [a,b] \quad (i=1,2), \tag{25}$$

$$\ell_1(x)\Big(\ell_1'(x)q'(x) + \ell_1(x)q''(x)\Big) \ge \frac{\lambda_1^2}{q^3(x)} - q(x)\ell_1(x) \qquad \text{for } x \in [a,b], \tag{26}$$

$$\ell_2(x)\Big(\ell_2'(x)q'(x) + \ell_2(x)q''(x)\Big) \le \frac{\lambda_2^2}{q^3(x)} - q(x)\ell_2(x) \qquad \text{for } x \in [a,b], \tag{27}$$

$$\ell_2(b) > y_1(\lambda_2). \tag{28}$$

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Example

Consider $q(x) = 2 - \cos x$ for $x \in [0, 2\pi]$.

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According to Corollary it is sufficient to find suitable functions $\ell_1(x)$ and $\ell_2(x)$ that satisfy (25)–(28). Obviously, we can start with positive constant functions. Then, if we put

$$\lambda_1^2 \stackrel{def}{=} \min\left\{(\ell_1^2 q''(x) + \ell_1 q(x))q^3(x) : x \in [0, \pi]\right\},$$
$$\lambda_2^2 \stackrel{def}{=} \max\left\{(\ell_2^2 q''(x) + \ell_2 q(x))q^3(x) : x \in [0, \pi]\right\},$$

we can easily verify that the inequalities (26) and (27) with $a = 0$, $b = \pi$ are fulfilled. Consequently, if also (28) is fulfilled, then we can conclude that (24) holds.

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Analyzing the function $x \mapsto (\ell^2 q''(x) + \ell q(x))q^3(x)$ in more details, one can show that the optimal values for constant functions ℓ_1 and ℓ_2 are

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Then, we get

$$\lambda_1^2 = \frac{540}{49} \approx 11.020408163, \qquad \lambda_2^2 = \frac{3125}{64} = 48.828125, \qquad y_1(\lambda_2) \approx 0.835507015894,$$

and we have the estimate

$$\frac{540}{49} \leq \lambda_0^2 \leq \frac{3125}{64}.$$

Let us pass to nonconstant functions $\ell_1(x)$ and $\ell_2(x)$. Then the choice

$$\ell_i(x) \stackrel{def}{=} a_i + b_i \cos x + c_i \sin x + d_i \sin x \cos x \qquad \text{for } x \in [0, \pi] \quad (i = 1, 2),$$

where

$$a_1 = 4.265, \qquad b_1 = 1.639, \qquad c_1 = -1.075, \qquad d_1 = -0.778,$$
$$a_2 = 3.605, \qquad b_2 = 1.025, \qquad c_2 = -0.408, \qquad d_2 = -0.222,$$

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guarantees that $\ell_1(x)$ and $\ell_2(x)$ satisfy (25)–(28) with $a = 0$, $b = \pi$, $\lambda_1^2 = 26.4$, and $\lambda_2^2 = 31.68$. Furthermore, note also that

$$\ell_1(\pi) < y_2(\lambda_1), \qquad y_2(\lambda_2) < \ell_2(\pi) \tag{29}$$

where $y_2(\lambda_i)$ is the greater root of (19) with $\lambda = \lambda_i$ $(i = 1, 2)$. Indeed,

$$\ell_1(\pi) = 2.626, \quad y_2(\lambda_1) \approx 2.62792828771, \quad y_2(\lambda_2) \approx 2.53762549442, \quad \ell_2(\pi) = 2.58$$

The condition (29) is stronger than (28) and allows strict inequalities in the threshold estimate. Therefore, according to Corollary we have

$$26.4 < \lambda_0^2 < 31.68.$$