Zero-convergent solutions for equations with generalized relativistic operator

Zuzana Došlá

Masaryk University, Brno, Czech Republic

Mauro Marini, Serena Matucci (University of Florence)

Czech-Georgian workshop, Brno, July 2–4, 2024

Table of Contents

Introduction

We study the existence of solutions for equation

$$(a(t)\Phi_R(x'))' + b(t)F(x) = 0, \quad t \in I = [t_0, \infty), \qquad (1)$$

satisfying the boundary conditions

$$x(t_0) = c > 0, \; x(t) > 0 \text{ and } x'(t) < 0 \text{ on } I, \; \lim_{t\to\infty} x(t) = 0, \qquad (2)$$

where $\Phi_R : (-1, 1) \to \mathbb{R}$ is generalized relativistic operator

$$\Phi_R(u) = \left(1 - |u|^{1+\alpha}\right)^{-\alpha/(1+\alpha)} |u|^\alpha \operatorname{sgn} u, \quad \alpha > 0.$$

• Special case $\alpha = 1$: $\Phi_M : (-1, 1) \to \mathbb{R}$ is Minkowski mean curvature operator (the relativity operator)

$$\Phi_M(u) = \frac{u}{\sqrt{1 - |u|^2}}, \qquad (3)$$

BVPs associated to equation (1):

P. Jebelean and C. Şerban, *Boundary value problems for discontinuous perturbations of singular Laplacian operator*, J. Math. Anal. Appl. **431** (2015).

P. Jebelean, J. Mawhin and C. Şerban, *A vector p-Laplacian type approach to multiple periodic solutions for the p-relativistic operator*, Commun. Contemp. Math. **19** (2017).

• The operator Φ_R occurs in studying some nonlinear elasticity problems

• The operator Φ_M occurs in studying certain extrinsic properties of the mean curvature of hypersurfaces in the relativity theory.

• The operator Φ_M can be found also in the theory of electromagnetism, where it is referred to as Born–Infeld operator.

A. Azzollini, *Ground state solutions for the Hénon prescribed mean curvature equation*, Adv. Nonlinear Anal. **8** (2019)

Z. Gao, S.B. Gudnason and Y. Yang, *Integer-squared laws for global vortices in the Born-Infeld wave equations*, Ann. Physics **400** (2019).

Global Kneser solutions and Minkowski mean curvature operator

Z. Došlá, M. Marini, S. Matucci, *Positive decaying solutions to BVPs with mean curvature operator*, Rend. Istit. Mat. Univ. Trieste **49** (2017):

$$(a(t)\Phi_M(x'))' + b(t)F(x) = 0, \tag{4}$$

where

$$\Phi_M(u) = \frac{u}{\sqrt{1-u^2}}, \qquad \int_{t_0}^{\infty} \frac{1}{a(t)}dt < \infty.$$

Asymptotic proximity between Kneser solutions of (4) and the corresponding ones of the linear equation

$$(a(t)y')' + b(t)y = 0$$

has been investigated.

Generalized relativistic operator

$$(a(t)\Phi_R(x'))' + b(t)F(x) = 0, \quad t \in I = [t_0, \infty), \qquad (1)$$

where

$$\Phi_R(u) = \left(1 - |u|^{1+\alpha}\right)^{-\alpha/(1+\alpha)} |u|^{\alpha} \operatorname{sgn} u, \quad \alpha > 0.$$

Functions a, b are continuous and positive on $[t_0, \infty)$, $t_0 \geq 0$, F is a continuous function on $\mathbb{R}$, $uF(u) > 0$ for $u \neq 0$.

The inverse operator Φ_R^* of Φ_R is

$$u = \Phi_R^*(z) = \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1/(\alpha+1)} \Phi_{1/\alpha}(z),$$

where $\Phi_{1/\alpha}(z) = |z|^{1/\alpha} \operatorname{sgn} z$.

Our goal:

Asymptotic proximity between Kneser solutions of (1) and the corresponding ones of the half-linear equation

$$\left(a(t)\,\Phi_\alpha(x')\right)' + b(t)\Phi_\alpha(x) = 0, \tag{5}$$

where

$$\Phi_\alpha(u) = |u|^\alpha \operatorname{sgn} u.$$

The inverse operator of Φ_α is

$$\Phi_{1/\alpha}(z) = |z|^{1/\alpha} \operatorname{sgn} z.$$

An abstract fixed point result

An approach for solving a BVP on the half-line $I = [t_0, \infty)$ is to reduce it to an abstract fixed point equation

$$x = \mathcal{T}(x), \tag{6}$$

where $\mathcal{T}$ is a possible nonlinear operator defined in a subset of a suitable Banach or Fréchet space X.

I is a noncompact interval: the choice as X of the Frechét space $C(I, \mathbb{R}^n)$ of the continuous vectors defined on I, endowed with the topology of uniform convergence on compact subsets of I appears to be the most suitable for verifying the compactness of $\mathcal{T}$.

Let $C(I, \mathbb{R}^2)$ be the Frechét space of the continuous vector functions $\underline{\mathbf{u}} = (u_1, u_2)$ defined on I, endowed with the topology of uniform convergence on compact subsets of I.

A subset Ω of $C(I, \mathbb{R}^2)$ is *bounded* if there exists a positive continuous function φ

$$|\underline{\mathbf{u}}(t)| \leq \varphi(t) \quad \text{for all } t \in I, \ \underline{\mathbf{u}} \in \Omega.$$

A set Ω is *relatively compact* in $C(I, \mathbb{R}^2)$ if it is bounded and the functions of Ω are equicontinuous on each compact subset of I.

Consider the differential equation

$$(a(t)\Phi(x'))' + F_1(t,x) = 0, \quad t \in I = [t_0, \infty), \tag{7}$$

where the operator $\Phi : I_\rho \to I_\sigma$ be an increasing odd homeomorphismus, $I_\rho = (-\rho, \rho)$, $I_\sigma = (-\sigma, \sigma)$, $0 < \rho \leq \infty$, $0 < \sigma \leq \infty$, the function $F_1 : I \times \mathbb{R} \to \mathbb{R}$ is continuous.
Set $G : I \times \mathbb{R} \to \mathbb{R}$ the continuous function

$$G(t,u)\Phi_\beta(u) = F_1(t,u).$$

Let Φ^* be its inverse

$$\Phi^*(z) = \Psi(z)\Phi_{1/\beta}(z),$$

where Ψ is a continuous positive function.

Theorem 1. *Let S_0 be a subset of $C(I, \mathbb{R}^2)$. Suppose that there exists a nonempty closed bounded convex subset $\Omega \subset C(I, \mathbb{R}^2)$ such that*

$$|v(t)| < \sigma\, a(t) \quad \text{for all } (u, v) \in \Omega \text{ and } t \in I,$$

and a nonempty closed subset S_1 of $S_0 \cap \Omega$ such that for each $(u, v) \in \Omega$ the half-linear equation

$$\frac{d}{dt}\left(H(t, v(t))\, \Phi_\beta(y')\right) + G(t, u(t))\, \Phi_\beta(y) = 0, \tag{8}$$

has a unique solution y_{uv} with $(y_{uv}, y_{uv}^{[1]}) \in S_1$,

$$H(t, v(t)) = a(t)\ \Psi^{-\beta}\left(\frac{v(t)}{a(t)}\right), \tag{9}$$

where $y_{uv}^{[1]}$ *is the quasiderivative of* y_{uv}, *that is*

$$y_{uv}^{[1]} = H(t, v(t))\,\Phi_\beta(y'_{uv}).$$

For each (u, v) *denote by* $\mathcal{T} : \Omega \to C(I, \mathbb{R}^2)$, *the operator given by*

$$\mathcal{T}(u, v) = (y_{uv}, y_{uv}^{[1]}). \tag{10}$$

Then $\mathcal{T}$ *has a fixed point* $(\widehat{x}, \widehat{y}) \in S_1 \subset S_0$ *such that* $\widehat{x}$ *is a solution of (7) and*

$$\widehat{y}(t) = a(t)\Phi(\widehat{x}'(t)).$$

Preliminaries on the half-linear equation

Consider the half-linear equation

$$\left(a(t)\,\Phi_\alpha(x')\right)' + b(t)\Phi_\alpha(x) = 0. \tag{11}$$

If (11) is nonoscillatory, then a nontrivial solution x_0 of (11) is said to be *the principal solution* of (11) if for every nontrivial solution x of (11) such that $x \neq \mu x_0$, $\mu \in \mathbb{R}$, the inequality

$$\frac{x_0'(t)}{x_0(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t.$$

holds.

The set of principal solutions of (11) is nonempty and principal solutions are determined up to a constant factor.
If x is a solution, we denote its *quasi-derivative* $x^{[1]}$ by

$$x^{[1]}(t) = a(t)\,\Phi_\alpha(x'(t)).$$

Positivity of principal solutions

The principal solution does not have zeros in a neighborhood of infinity. The positiveness of the principal solution on an *a-priori* closed fixed unbounded interval $[T, \infty)$, $T \geq 1$, is a more subtle question.

Consider the half-linear equations

$$\big(a_1(t)\,\Phi_\alpha(z')\big)' + b_1(t)\,\Phi_\alpha(z) = 0, \tag{12}$$

and

$$\big(a_2(t)\,\Phi_\alpha(w')\big)' + b_2(t)\,\Phi_\alpha(w) = 0, \tag{13}$$

where $a_i, b_i, i = 1, 2$, are positive continuous functions for $t \geq t_0$ such that

$$a_2(t) \leq a_1(t)\ , \quad b_2(t) \geq b_1(t). \tag{14}$$

Equation (13) is a *majorant* of (12) and, analogously, (12) is a *minorant* of (13).

A comparison result

Lemma

Assume that (13) is nonoscillatory and (14) is valid. Denote by z_0 and w_0 the principal solutions of (12) and (13), respectively. If w_0 does not have zeros on $[T,\infty)$, then the following holds.
(j_1) The principal solution z_0 does not have zeros on $[T,\infty)$.
(j_2) We have for $t \geq T$

$$\frac{z_0^{[1]}(t)}{\Phi_\alpha(z_0(t))} \leq \frac{w_0^{[1]}(t)}{\Phi_\alpha(w_0(t))},$$

where $z_0^{[1]}$ is the quasi-derivative of z_0 and $w_0^{[1]}$ is the one of w_0.

Global Kneser solutions

Consider equation

$$(a(t)\Phi_R(x'))' + b(t)F(x) = 0, \quad t \in I = [t_0, \infty), \tag{1}$$

and the boundary conditions

$$x(t_0) = c > 0, \ x(t) > 0 \text{ and } x'(t) < 0 \text{ on } I, \ \lim_{t\to\infty} x(t) = 0, \tag{2}$$

where $\Phi_R : (-1, 1) \to \mathbb{R}$ is generalized relativistic operator

$$\Phi_R(u) = \left(1 - |u|^{1+\alpha}\right)^{-\alpha/(1+\alpha)} \Phi_\alpha(u), \quad \alpha > 0.$$

Assume

$$\inf_{t\geq t_0} a^{1/\alpha}(t) \int_t^\infty a^{-1/\alpha}(s)ds = \lambda > 0,$$

$$\lim_{u\to 0+} \frac{F(u)}{u^\alpha} = F_0, \quad 0 \leq F_0 < \infty.$$

Define

$$M = \sup_{u\in(0,\lambda]} \frac{F(u)}{u^\alpha}.$$

Theorem 2. *Assume*

$$Y_1 = \int_{t_0}^{\infty} b(t) \left(\int_t^{\infty} a^{-1/\alpha}(s)\ ds \right)^{\alpha}\ dt < \infty,$$

$$J_1 = \int_{t_0}^{\infty} a^{-1/\alpha}(t) \left(\int_{t_0}^{t} b(s)\ ds \right)^{1/\alpha}\ dt < \infty\,.$$

If the half-linear equation

$$\left(a(t)\Phi_\alpha(z')\right)' + M\, b(t)\ \Phi_\alpha(z) = 0, \quad t \geq t_0,$$

is nonoscillatory and its principal solution z_0 is positive decreasing on $I = [t_0, \infty)$, then for any constant c such that

$$0 < c < \lambda$$

(1) has a solution x satisfying the boundary conditions (2).

An example of a suitable half-linear equation can be obtained using the half-linear Euler differential equation

$$\left(\Phi_\beta(x')\right)' + \left(\frac{\beta}{\beta+1}\right)^{\beta+1} t^{-\beta-1}\Phi_\beta(x) = 0, \qquad t \ge t_0 > 0. \quad (15)$$

It is known that (15) is nonoscillatory and the function

$$x_0(t) = t^{\beta/(\beta+1)}$$

is the principal solution of (15). The change of variable

$$y = \Phi_\beta(x')$$

transforms (15) into the reciprocal equation

$$\left(t^{(\beta+1)/\beta}\Phi_{1/\beta}(y')\right)' + \left(\frac{\beta}{\beta+1}\right)^{(\beta+1)/\beta} \Phi_{1/\beta}(y) = 0, \qquad t \ge t_0 > 0,$$

Corollary 1

Assume $Y_1 < \infty$ and $J_1 < \infty$. If

$$a(t) \geq t^{1+\alpha} \quad \text{and} \quad M\, b(t) \leq \left(\frac{1}{1+\alpha}\right)^{1+\alpha},$$

is satisfied for $t \geq t_0 > 0$,, then for any constant c, such that $0 < c < \lambda$, equation (1) has a solution x satisfying the boundary conditions (2).

Necessary condition

Theorem 3. *If*

$$\int_{t_0}^{\infty} \Phi_R^*\left(\frac{k}{a(s)}\right) ds = \infty \quad \textit{for any positive constant } k,$$

where

$$\Phi_R^*(z) = \left(1 + |z|^{(\alpha+1)/\alpha}\right)^{-1/(\alpha+1)} |z|^{1/\alpha} \operatorname{sgn} z,$$

then (1) does not have solutions x satisfying (2).

References

- Cecchi M., Furi M., Marini M.: *On continuity and compactness of some nonlinear operators associated with differential equations in noncompact intervals,* Nonlinear Anal. **9** (1985).
- Došlá Z., Marini M., Matucci S.: *Positive decaying solutions to BVPs with mean curvature operator*. Rend. Istit. Mat. Univ. Trieste Vol. 49 (2017).
- Z. Došlá, M. Marini and S. Matucci, *On unbounded solutions for differential equations with mean curvature operator*, Czech. Math. J. (2023). https://doi.org/10.21136/CMJ.2023.0111-23
- Z. Došlá, M. Marini and S. Matucci, *Zero-convergent solutions for equations with generalized relativistic operator: a fixed point approach*, submitted for publication.