On Bounded Monotone Solutions to Singular in the Time Variable Two-Dimensional Differential Systems

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In the present report, for singular in the time variable differential systems

$$u_i' = f_i(t, u_1, u_2) \;\; (i = 1, 2), \tag{1}$$

$$u_i' = g_i(t, u_{3-i}) \;\; (i = 1, 2), \tag{1'}$$

we consider the problem on the existence of so-called Kneser solution, defined on the positive half-axis, and satisfying the initial condition

$$\lim_{t\to 0} u_1(t) = c.$$

Conditions that are unimprovable in a certain sense are given, guaranteeing, respectively, the existence, uniqueness, monotonicity, boundedness, and vanishness at infinity of a solution to this problem.

We use the following notation: $\mathbb{R} = \{-\infty, +\infty\}$, $\mathbb{R}_+ = [0, +\infty[\,$, $\mathbb{R}_- =]-\infty, 0]$.

$L_{loc}(I)$, where I is either open or semi-open interval, is the space of defined in I real functions whose restrictions to any closed bounded interval contained in I are Lebesgue integrable.

We say that the function $f : I \times \mathbb{R}^m \to \mathbb{R}$ belongs to the Carathéodory space $K_{loc}(I \times \mathbb{R}^m)$ if the function $f(\cdot, x_1, \dots, x_m) : I \to \mathbb{R}$ is measurable for any arbitrarily fixed $(x_1, \dots, x_m) \in \mathbb{R}^m$, the function $f(t, \cdot, \dots, \cdot) : \mathbb{R}^m \to \mathbb{R}$ is continuous for almost all arbitrarily fixed $t \in I$, and

$$\max\Big\{|f(\cdot, x_1, \dots, x_m)| : \sum_{i=1}^m |x_i| \le x\Big\} \in L_{loc}(I)$$

for every positive constant x.

We investigate systems (1) and (1′) in the case, where $f_1 \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}^2)$, $f_2 \in K_{loc}(]0, +\infty[\, \times \mathbb{R}^2)$, and the conditions

$$f_i(t, 0, 0) = 0, \;\; (-1)^i f_i(t, x_1, x_2) \ge 0 \;\text{ for } t > 0, \; x_1 \in \mathbb{R}_+, \; x_2 \in \mathbb{R}_- \;(i = 1, 2),$$

$$g_1 \in K_{loc}(\mathbb{R}_+ \times \mathbb{R}), \;\; g_2 \in K_{loc}(]0, +\infty[\, \times \mathbb{R}),$$

$$g_i(t, y) \ge g_i(t, x), \;\; g_i(t, -x) = -g_i(t, x) \;\text{ for }\; t > 0, \;\; y \ge x \;\; (i = 1, 2)$$

are satisfied. At the same time, we do not exclude cases where the differential systems under consideration have non-integrable singularities in the time variable at the point $t = 0$, namely, the cases, where

$$\int_0^1 f_2(t, x_1, x_2)\, dt = +\infty, \;\; \int_0^1 g_2(t, x_1)\, dt = +\infty \;\text{ for }\; x_1 > 0, \;\; x_2 < 0.$$