The Inverse Problem for Periodic Travelling Waves of the Linear 1D Shallow-Water Equations

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The motion of small amplitude waves of a water layer with variable depth along the x-axis is described by the equations of the shallow water theory

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\,[h(x)u] = 0, \qquad \frac{\partial u}{\partial t} + g\frac{\partial \eta}{\partial x} = 0, \tag{1}$$

where $\eta(x,t)$ is the vertical water surface elevation, $u(x,t)$ is the depth-averaged water flow velocity (also called wave velocity), $h(x)$ is the unperturbed water depth and g is the gravity acceleration. In what follows, we assume without loss of generality that $g = 1$.

The shallow water equations conform a system of coupled PDEs of first order that can be easily decoupled into a single wave equation for the surface displacement

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{\partial}{\partial x}\left[h(x)\frac{\partial \eta}{\partial x}\right] = 0, \tag{2}$$

or for the wave velocity

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2}{\partial x^2}\,[h(x)u] = 0. \tag{3}$$

There is a considerable number of papers devoted to finding sufficient conditions on the bottom profile $h(x)$ to ensure the existence of travelling waves or other explicit solutions. A travelling wave is a special solution of the form $q(x)\exp i\,[\omega t - \Psi(x)]$, where both q and Ψ are scalar real-valued functions. In the related literature, $q(x)$ is known as the amplitude of the travelling wave, ω is the frequency and $\Psi(x)$ is the phase, which is called non-trivial if it is non-constant. In this talk, we are going to study the following inverse problem: *given a prescribed amplitude $q(x)$, can we determine a suitable bottom profile $h(x)$ allowing the equation to admit a travelling wave with amplitude $q(x)$?*