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The Dirichlet Problem for Second Order Singular Differential Equations with Deviating Arguments

Nino Partsvania[1,2]

[1]*Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University*
[2]*International Black Sea University, Tbilisi, Georgia*

nino.partsvania@tsu.ge

On a finite interval $]a, b[$, we consider the linear and nonlinear differential equations

$$u''(t) = p(t)u(\tau(t)) + q(t) \tag{1}$$

and

$$u''(t) = f(t, u(\tau(t))) \tag{2}$$

with the Dirichlet weighted boundary conditions

$$\lim_{t\to a} \frac{u(t)}{(t-a)^{\alpha}} = 0, \quad \lim_{t\to b} \frac{u(t)}{(b-t)^{\beta}} = 0. \tag{3}$$

Here, α and β are constants satisfying the inequalities

$$0 \le \alpha < 1, \quad 0 \le \beta < 1,$$

p and q $:]a, b[\to \mathbb{R}$ are the Lebesgue integrable on every closed interval contained in $]a, b[$ functions, $\tau : [a, b] \to [a, b]$ is a measurable function, and f $:]a, b[\times\mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathèodory conditions.

Eq. (1) is said to be regular if p and q are integrable on $[a, b]$ functions, and it is said to be singular if

$$\int_a^b (|p(t)| + |q(t)|)dt = +\infty.$$

As for Eq. (2), it is said to be regular if

$$\int_a^b f^*(t, x)dt < +\infty \ \text{ for } \ x > 0,$$

and it is said to be singular in the time variable if

$$\int_a^b f^*(t,x)dt = +\infty \ \text{ for } \ x > 0,$$

where

$$f^*(t,x) = \max\{|f(t,y)| : |y| \le x\}.$$

For regular equations of the type (1) and (2), the Dirichlet problem is investigated in detail (see [3, 5] and the references therein). Sufficient conditions for the solvability and unique solvability of the above mentioned problem for singular functional differential equations (including Eq. (1) and Eq. (2)) are contained in [1, 2, 4, 6].

We have established new conditions guaranteeing respectively the solvability and unique solvability of the singular problems (1), (3) and (2), (3). In contrast to the results from [1, 2, 4, 6], they cover the cases where the functions p and f have singularities of arbitrary order at $t = a$ and $t = b$.

Below we use the following notation.

$$[x]_- = \frac{|x| - x}{2},$$

$$\chi(t) = \begin{cases} 1 & \text{if } \ t = \tau(t), \\ 0 & \text{if } \ t \ne \tau(t), \end{cases}$$

$$\ell_{\alpha,\beta}(p)(t) = \chi(t)(t-a)^\alpha(b-t)^\beta[p(t)]_- + (1-\chi(t))(\tau(t)-a)^\alpha(b-\tau(t))^\beta|p(t)|.$$

It is obvious that if g is a nonnegative function, then

$$\ell_{\alpha,\beta}(-g)(t) = \big[\chi(t)(t-a)^\alpha(b-t)^\beta + (1-\chi(t))(\tau(t)-a)^\alpha(b-\tau(t))^\beta\big]\, g(t).$$

Along with (1) we have to consider the corresponding homogeneous differential equation

$$u''(t) = p(t)u(\tau(t)). \tag{1_0}$$

The following theorem is valid.

Theorem 1. *If*

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}\ell_{\alpha,\beta}(p)(t)dt < +\infty$$

and

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}|q(t)|dt < +\infty, \tag{4}$$

then problem (1), (3) *is uniquely solvable if and only if the corresponding homogeneous problem* (1_0), (3) *has only the trivial solution.*

Corollary 1. *If*

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}\ell_{\alpha,\beta}(p)(t)\,dt \le b-a \tag{5}$$

and condition (4) *is satisfied, then problem* (1), (3) *has one and only one solution.*

Remark 1. Inequality (5) in Corollary 1 is unimprovable and it cannot be replaced by the inequality

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}\ell_{\alpha,\beta}(p)(t)\,dt \le b-a+\varepsilon,$$

no matter how small $\varepsilon > 0$ is.

Example 1. Let $a < a_0 < b_0 < b$, $\alpha = \beta$,

$$\tau(t) = t, \quad p(t) = \exp\bigg(\frac{1}{(t-a)(b-t)}\bigg) \ \text{ for } \ t \in [a,a_0] \cup [b_0,b],$$

and let the restriction of p to $[a_0,b_0]$ be an integrable function such that

$$\int_{a_0}^{b_0} |p(s)|\,ds \le \frac{4}{b-a}.$$

If, moreover, condition (4) is satisfied, then by virtue of Corollary 1 problem (1), (3) has one and only one solution.

This example shows that under the conditions of Theorem 1 and Corollary 1 the function p may have singularities of arbitrary order at $t = a$ and $t = b$.

Theorems below on the existence and uniqueness of a solution of problem (2), (3) concern the cases where in the domain $]a,b[\times\mathbb{R}$ one of the following two conditions is fulfilled:

$$\chi(t)f(t,x)\operatorname{sgn}(x) - (1-\chi(t))|f(t,x)| \ge -g(t)|x| - h(t), \tag{6}$$

$$\chi(t)\,[f(t,x)-f(t,y)]\operatorname{sgn}(x-y) - (1-\chi(t))|f(t,x)-f(t,y)| \ge -g(t)|x-y|, \tag{7}$$

where g and $h :]a,b[\to [0,+\infty[$ are integrable on every closed interval contained in $]a,b[$ functions.

Theorem 2. *If along with* (6) *the conditions*

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}\ell_{\alpha,\beta}(-g)(t)\,dt < b-a, \tag{8}$$

$$\int_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}h(t)\,dt < +\infty \tag{9}$$

hold, then problem (2), (3) *has at least one solution.*

Theorem 3. *If conditions* (7)–(9) *are fulfilled, where*

$$h(t) = |f(t,0)|,$$

then problem (2), (3) *has one and only one solution.*

Remark 2. Inequality (8) in Theorems 2 and 3 cannot be replaced by the inequality

$$\int\limits_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}\ell_{\alpha,\beta}(-g)(t)\,dt \le b-a+\varepsilon,$$

where $\varepsilon > 0$. However, the question of whether it is possible to replace (8) by the nonstrict inequality

$$\int\limits_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}\ell_{\alpha,\beta}(-g)(t)\,dt \le b-a$$

remains open.

Example 2. Let $a < a_0 < b_0 < b$,

$$\tau(t) = t, \quad f(t,x) = \exp\left(\frac{1+|x|}{(t-a)(b-t)}\right)x^{2m-1} + q(t) \ \text{ for } \ t \in [a,a_0]\cup[b_0,b], \quad x \in \mathbb{R},$$
$$f(t,x) = p(t)|x| + q(t) \ \text{ for } \ t \in [a_0,b_0], \quad x \in \mathbb{R},$$

where m is a natural number, $p : [a_0,b_0] \to \mathbb{R}$ and $q :]a,b[\to \mathbb{R}$ are measurable functions. If, moreover,

$$\int\limits_{a_0}^{b_0} |p(s)|\,ds < \frac{4}{b-a}$$

and

$$\int\limits_a^b (t-a)^{1-\alpha}(b-t)^{1-\beta}|q(t)|\,dt < +\infty,$$

then by Theorem 3 problem (2), (3) has one and only one solution.

Consequently, under the conditions of Theorems 2 and 3 the function f may have singularities of arbitrary order at $t = a$ and $t = b$. Moreover, the function f may satisfy also the condition

$$\lim_{|x|\to+\infty} \frac{f(t,x)}{x} = +\infty \ \text{ for } \ t \in I_0,$$

where

$$I_0 = \{t \in]a,b[\,:\, \tau(t) = t\}.$$

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