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On Rapidly Growing Solutions of Second Order Nonlinear Differential Equations

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Let $\mathbb{R}_+ = [0, +\infty[$, and $f : \mathbb{R}_+ \times \mathbb{R}_+^2 \to \mathbb{R}_+$ be a continuous function, satisfying the local Lipschitz condition with respect to the last two arguments.

Consider the differential equation

$$u'' = f(t, u, u'). \tag{1}$$

A solution of this equation on an arbitrary interval $I \subset \mathbb{R}_+$ is sought on the set of twice continuously differentiable on I functions, satisfying the inequalities

$$u(t) \geq 0, \ \ u'(t) \geq 0 \ \text{ for } \ t \in I.$$

A nontrivial solution of Eq. (1), defined on some infinite interval $[a, +\infty[\subset \mathbb{R}_+$, is said to be **proper**. It is obvious that an arbitrary proper solution u of Eq. (1) is nondecreasing together with its first derivative and satisfies one of the following two conditions:

$$\lim_{t\to+\infty} u'(t) < +\infty,$$
$$\lim_{t\to+\infty} u'(t) = +\infty.$$

In the first case, the solution u is said to be **slowly growing**, and in the second case, it is said to be **rapidly growing**.

A solution u of Eq. (1), defined on some finite interval $[a, b[\subset \mathbb{R}_+$, is said to be **blow-up** if

$$\lim_{t\to b} u'(t) = +\infty.$$

R. Emden and R. H. Fowler have investigated in detail asymptotic properties of proper and blow-up monotone solutions of the frequently occurring in applications differential equation

$$u'' = t^\sigma u^\lambda.$$

The results obtained by them are reflected in the well-known monograph by R. Bellman ([2], Ch. VII). The analogous theory for the Emden-Fowler differential equation

$$u'' = p(t)u^{\lambda}$$

with the coefficient $p : \mathbb{R}_+ \to \mathbb{R}_+$ of general type was constructed by I. Kiguradze [8] (see, also [13], Ch. V).

The foundations of the asymptotic theory of monotone solutions of arbitrary order nonlinear differential equations were laid back in the late sixties of the last century and it still remains relevant (see [1], [3]-[7], [9]-[14], and the references therein).

On the basis of the method proposed in [14], we established new necessary and sufficient conditions for Eq. (1) to have rapidly growing solutions and obtained two-sided asymptotic estimates of these solutions. Below we give exactly these results.

We study the cases, where the function f satisfies either the inequality

$$f(t,x,y) \geq \varphi(t,x) \ \text{ for } \ t \geq a, \ \ 0 \leq x \leq ty, \tag{2}$$

or the inequality

$$f(t,x,y) \leq \psi(t,x) \ \text{ for } \ t \geq a, \ \ 0 \leq x \leq ty, \tag{3}$$

where a is a positive number, and $\varphi : [a,+\infty[\, \times \mathbb{R}_+ \to \mathbb{R}_+$ and $\psi : [a,+\infty[\, \times \mathbb{R}_+ \to \mathbb{R}_+$ are continuous functions.

We put

$$\varphi_0(t,x) = \Big(\int_0^x \varphi(t,s)\,ds \Big)^{\frac{1}{2}} \ \text{ for } \ t \geq a, \ \ x \geq 0,$$

and consider the differential equations

$$\frac{dv}{dt} = \varphi_0(t,v), \tag{4_1}$$

$$\frac{dv}{dt} = \psi(t,v), \tag{4_2}$$

with the limit condition

$$\lim_{t\to+\infty} v(t) = +\infty. \tag{5}$$

A solution v^* of problem $(4_1),(5)$ (a solution v_* of problem $(4_2),(5)$), defined on the interval $[a,+\infty[$, is said to be **upper** (**lower**) if for any $t_0 \in [a,+\infty[$ an arbitrary solution v of problem $(4_1),(5)$ (of problem $(4_2),(5)$), defined on the interval $[t_0,+\infty[$, admits the estimate

$$v(t) \leq v^*(t) \ \ \big(v(t) \geq v_*(t)\big) \ \text{ for } \ t_0 \leq t < +\infty.$$

Theorem 1. *Let inequality* (2) *be satisfied, the function* φ_0 *do not increase in the first argument,*

$$\lim_{t\to+\infty} \varphi_0(t,x) = 0 \ \textit{ for } \ x > 0, \tag{6}$$

and let problem $(4_1),(5)$ *on the interval* $[a,+\infty[$ *have an upper solution* v^*. *Then the differential equation* (1) *has a two-parametric set of blow-up solutions, and for the existence of a rapidly growing solution, it is necessary that the equality*

$$\lim_{t\to+\infty}\frac{v^*(t)}{t}=+\infty \tag{7}$$

be satisfied.

Theorem $1'$. *Let conditions* (2), (6) *hold, the function* φ_0 *do not increase in the first argument, and let problem* $(4_1),(5)$ *on the interval* $[a,+\infty[$ *have an upper solution* v^*, *satisfying equality* (7). *Then an arbitrary rapidly growing solution* u *of Eq.* (1) *in some neighborhood of* $+\infty$ *admits the estimate*

$$u(t)\le v^*(t).$$

Corollary 1. *Let the inequality*

$$f(t,x,y)\ge p(t)g(x) \ \text{ for } \ t\ge a, \ \ 0\le x\le ty$$

hold, where $p:[a,+\infty[\,\to\mathbb{R}_+$ *is a continuous nonincreasing function, while* $g:\mathbb{R}_+\to\mathbb{R}_+$ *is a continuous function such that*

$$0<\int_t^{+\infty}\sqrt{p(s)}\,ds<+\infty \ \text{ for } \ t\ge a,$$

$$g_0(x)=\Big(\int_0^x g(s)\,ds\Big)^{\frac12}>0, \ \ \int_0^{+\infty}\frac{ds}{g_0(s)}<+\infty \ \text{ for } \ x>0.$$

If, moreover,

$$\lim_{t\to+\infty}\frac{g^*\Big(\int_t^{+\infty}\sqrt{p(s)}\,ds\Big)}{t}=+\infty,$$

where $g^*:]0,+\infty[\,\to\,]0,+\infty[$ *is a function defined from the equality*

$$\int_{g^*(x)}^{+\infty}\frac{ds}{g_0(s)}=x \ \text{ for } \ x>0,$$

then Eq. (1) *has a two-parametric set of blow-up solutions, and an arbitrary rapidly growing solution of that equation in some neighborhood of* $+\infty$ *admits the estimate*

$$u(t)\le g^*\Big(\int_t^{+\infty}\sqrt{p(s)}\,ds\Big).$$

Consider now the question on the lower estimate of rapidly growing solutions of Eq. (1).

The following theorem is valid.

Theorem 2. *Let inequality* (3) *hold, the function* ψ *do not decrease in the second argument, and let problem* (4_2), (5) *on the interval* $[a,+\infty[$ *have a lower solution* v_*. *Then an arbitrary rapidly growing solution* u *of Eq.* (1) *on some interval* $[t_0,+\infty[\subset [a,+\infty[$ *admits the estimate*

$$u(t) > \int_{t_0}^{t} v_*(s)ds \ \ \text{for} \ \ t\geq t_0.$$

Corollary 2. *Let the inequality*

$$f(t,x,y)\leq q(t)h(y) \ \ \text{for} \ \ t\geq a, \ \ 0\leq x\leq ty$$

hold, where $q:[a,+\infty[\to\mathbb{R}_+$ *is a continuous function, while* $h:[a,+\infty[\to\mathbb{R}_+$ *is a continuous nondecreasing function such that*

$$h(y)>0, \ \ \int_{y}^{+\infty}\frac{ds}{h(s)}<+\infty \ \text{for} \ \ y>0,$$
$$0<\int_{t}^{+\infty} q(s)ds<r \ \ \text{for} \ \ t\geq a,$$

where

$$r=\int_{0}^{+\infty}\frac{ds}{h(s)}.$$

Then an arbitrary rapidly growing solution of Eq. (1) *on some interval* $[t_0,+\infty[\subset [a,+\infty[$ *admits the estimate*

$$u(t) > \int_{t_0}^{t} h_*\Big(\int_{s}^{+\infty} q(\tau)d\tau\Big)ds \ \ \text{for} \ \ t\geq t_0,$$

where h_* *is a function defined from the equality*

$$\int_{h_*(y)}^{+\infty}\frac{ds}{h(s)}=y \ \ \text{for} \ \ 0<y<r.$$

The above theorems and their corollaries leave open the question of the existence of a rapidly growing solution of Eq. (1). Estimates for a rapidly growing solution appearing in these statements are obtained under the a priori assumption that such a solution exists. The answer to this question is given by the following theorem.

Theorem 3. *Let the inequalities*

$$\varphi(t,x)\leq f(t,x,y)\leq \psi(t,y) \ \ \text{for} \ \ t\geq a, \ \ 0\leq x\leq ty$$

hold, where $\varphi:[a,+\infty[\times\mathbb{R}_+\to\mathbb{R}_+$ *is a nonincreasing in the first argument continuous function, satisfying condition* (6), *while* $\psi:[a,+\infty[\times\mathbb{R}_+\to\mathbb{R}_+$ *is a nondecreasing in the second argument function, satisfying the condition*

$$\int_{a}^{+\infty}\psi(s,y)ds<+\infty \ \ \text{for} \ \ y>0.$$

If, moreover, problem $(4_1), (5)$ *on the interval* $[a,+\infty[$ *has an upper solution* v^**, and problem* $(4_2), (5)$ *on the interval* $[a,+\infty[$ *has a lower solution* v_**, then Eq.* (1) *has a two-parametric set of blow-up solutions, a one-parametric set of rapidly growing solutions, and an arbitrary rapidly growing solution* u *on some interval* $[t_0,+\infty[\subset[a,+\infty[$ *admits the estimates*

$$\int_{t_0}^{t} v_*(s)ds < u(t) \leq v^*(t) \ \text{ for } \ t \geq t_0.$$

Corollary 3. *Let the inequalities*

$$p(t)g(x) \leq f(t,x,y) \leq q(t)h(y) \ \text{ for } \ t \geq a, \ \ 0 \leq x \leq ty$$

hold, where p *and* g *are functions satisfying the conditions of Corollary 1, while* q *and* h *are functions satisfying the conditions of Corollary 2. Then Eq.* (1) *has a two-parametric set of blow-up solutions, a one-parametric set of rapidly growing solutions, and an arbitrary rapidly growing solution* u *on some interval* $[t_0,+\infty[\subset[a,+\infty[$ *admits the estimates*

$$\int_{t_0}^{t} h_*\Big(\int_{s}^{+\infty} q(\tau)d\tau\Big)ds < u(t) \leq g^*\Big(\int_{t}^{+\infty} \sqrt{p(s)}ds\Big) \ \text{ for } \ t \geq t_0.$$

As an example, consider the differential equation

$$u'' = t^{-\mu}u^{\lambda}(u')^{\lambda_1}, \tag{8}$$

where

$$\mu > 0, \ \ \lambda \geq 0, \ \ \lambda_1 \geq 0, \ \ \lambda + \lambda_1 > 1.$$

In the case, where $\mu > \lambda + 1$, we put

$$\ell_1 = \frac{\lambda+\lambda_1-1}{2(\mu+\lambda_1-1)}\Big(\frac{\mu-\lambda-1}{\lambda+\lambda_1-1}\Big)^{\frac{1}{\lambda+\lambda_1-1}}, \ \ \ell_2 = \left(\frac{2}{\lambda+\lambda_1-1}\Big(\frac{\lambda+\lambda_1+1}{2}\Big)^{\frac{1}{2}}\right)^{\frac{2}{\lambda+\lambda_1-1}}.$$

Corollary 3 yields the following

Corollary 4. *If* $\mu \leq \lambda + 1$*, then all nontrivial solutions of Eq.* (8) *are blow-up, and if*

$$\mu > \lambda + 1,$$

then Eq. (8) *has a two-parametric set of blow-up solutions, a one-parametric set of rapidly growing solutions, and an arbitrary rapidly growing solution* u *in some neighborhood of* $+\infty$ *admits the estimates*

$$\ell_1 t^{\frac{\mu+\lambda_1-2}{\lambda+\lambda_1-1}} \leq u(t) \leq \ell_2 t^{\frac{\mu+\lambda_1-2}{\lambda+\lambda_1-1}}.$$

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