On the Conditions of Solvability of the Cauchy Problem for Singular in Time and Phase Variables Differential Equations with Deviating Arguments

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The Cauchy problem for ordinary differential equations and functional differential equations with nonintegrable singularities in the time variable has been studied in sufficient detail (see [1, 2, 3, 4] and the references therein). I. Kiguradze in [5] established optimal sufficient conditions for the solvability of that problem for singular in phase variables higher order differential equations. In [6], unimprovable in a certain sense conditions are found guaranteeing, respectively, the solvability and unique solvability of the Cauchy weighted problem for singular in time and phase variables higher order delay differential equations.

After the publication of [6], together with I. Kiguradze, we studied the Cauchy problem for singular differential equations with deviating arguments in the cases where these deviations or some of them are not delay. Therefore we discuss the global solvability of the problem under consideration.

In the present report, for the sake of simplicity of formulation, our joint with I. Kiguradze results are given not for general differential equations but only for equations of the following types

$$u^{(n)}(t)=p(t)u^{\lambda}(\eta(t))+q(t)u^{-\mu}(\tau(t))+r(t) \tag{1}$$

and

$$u^{(n)}(t)=p(t)u^{\lambda}(\eta(t))+q(t)u^{-\mu}(\tau(t)). \tag{1_0}$$

We study problems on the existence of solutions of these equations under the initial conditions

$$u^{(i-1)}(0)=0 \quad (i=1,\dots,n). \tag{2}$$

Here $n\geq 1$, $\lambda\in\,]0,1]$, $\mu>0$, $a>0$, $p:\,]0,a[\,\to[0,+\infty[$ is a function which is integrable on the interval $]\varepsilon,a[$ for any $\varepsilon\in\,]0,a[$, q and $r:\,]0,a[\,\to[0,+\infty[$ are integrable on $]0,a[$ functions, and the deviations η and $\tau:\,]0,a[\,\to\,]0,a]$ are measurable functions. Moreover, it is assumed that

$$\int\limits_0^t q(s)\,ds>0,\quad \int\limits_0^t r(s)\,ds>0 \ \text{ for } \ 0<t\leq a.$$

We use the following notation.

$\widetilde{C}^{n-1}([0,a])$ is the space of $(n-1)$-times continuously differentiable on $[0,a]$ real functions whose $(n-1)$th order derivative is absolutely continuous;

$$q_n(t)=\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1}q(s)ds, \quad r_n(t)=\frac{1}{(n-1)!}\int\limits_0^t (t-s)^{n-1}r(s)ds.$$

Solutions of problems $(1),(2)$ and $(1_0),(2)$ are sought on the set

$$\big\{u\in \widetilde{C}^{n-1}([0,a]):\ u(t)>0 \text{ for } 0<t\le a\big\}.$$

Theorem 1. *Let the conditions*

$$\eta(t)\le t \quad \textit{for} \ \ 0<t<a, \tag{3}$$

$$\int\limits_0^a \eta^{(n-1)\lambda}(s)p(s)ds<+\infty \tag{4}$$

hold, or let condition (3) *be violated and instead of* (4) *the inequality*

$$\int\limits_0^a \eta^{(n-1)\lambda}(s)p(s)ds<((n-1)!)^{\lambda} \tag{4'}$$

be satisfied. If, moreover, the function q *satisfies the condition*

$$\int\limits_0^a r_n^{-\mu}(\tau(s))q(s)ds<+\infty, \tag{5}$$

then problem $(1),(2)$ *has at least one solution.*

Theorem 2. *Let the conditions*

$$\eta(t)\le t, \ \ \tau(t)\le t \ \ \textit{for} \ \ 0<t<a, \tag{6}$$

$$\int\limits_0^a \Big(\eta^{n-1}(s)r_n^{\lambda-1}(\eta(s))p(s)+\tau^{n-1}(s)r_n^{-1-\mu}(\tau(s))q(s)\Big)ds<+\infty \tag{7}$$

hold, or let at least one of the inequalities in (6) *be violated and instead of* (7) *the inequality*

$$\int\limits_0^a \Big(\lambda\eta^{n-1}(s)r_n^{\lambda-1}(\eta(s))p(s)+\mu\tau^{n-1}(s)r_n^{-1-\mu}(\tau(s))q(s)\Big)ds<(n-1)! \tag{7'}$$

be satisfied. Then problem $(1),(2)$ *has one and only one solution.*

Remark 1. Let $p_0 :]0,a[\to [0,+\infty[$ and $q_0 :]0,a[\to [0,+\infty[$ be arbitrary integrable functions, and

$$p(t) \equiv p_0(t)\eta^{(1-n)\lambda}(t), \quad q(t) \equiv q_0(t) r_n^{\mu}(\tau(t)).$$

Then, by Theorem 1, for the solvability of problem (1), (2), it suffices that either inequality (3) holds, or the inequality

$$\int\limits_0^a p_0(s)\,ds < ((n-1)!)^{\lambda} \tag{8}$$

is satisfied. In particular, if $n > 1$, $\eta(t) \equiv a\exp\big(-\frac{a}{t}\big)$, and

$$\liminf_{t\to 0} p_0(t) > 0, \tag{9}$$

then problem (1), (2) is solvable and the function $t \mapsto p(t)$ at the point $t = 0$ has a singularity of infinity order. And if

$$n > 2, \ \ 1/(n-1) < \alpha < 1, \ \ \eta(t) \equiv a^{1-\alpha} t^{\alpha},$$

and inequalities (8) and (9) are satisfied, then problem (1), (2) is still solvable and the function $t \mapsto p(t)$ at the point $t = 0$ has a singularity of order $(n-1)\alpha$.

Remark 2. Let $p_0 :]0,a[\to [0,+\infty[$ and $q_0 :]0,a[\to]0,+\infty[$ be arbitrary integrable functions, and

$$p(t) \equiv p_0(t)\eta^{1-n}(t) r_n^{1-\lambda}(\eta(t)), \quad q(t) \equiv q_0(t)\tau^{1-n}(t) r_n^{1+\mu}(\tau(t)).$$

Then, by Theorem 2, for the unique solvability of problem (1), (2), it suffices that either inequalities (6) hold, or the inequality

$$\int\limits_0^a \big(\lambda p_0(s) + \mu q_0(s)\big)\,ds < (n-1)!$$

is satisfied. In particular, if $n > 1$, $\lambda \in]\frac{1}{n}, 1]$, $\eta(t) = a\exp\big(-\frac{a}{t}\big)$, $\tau(t) \le t$ for $0 < t < a$, and

$$\liminf_{t\to 0} p_0(t) > 0, \quad \liminf_{t\to 0} r(t) > 0,$$

then problem (1), (2) is uniquely solvable and the function $t \mapsto p(t)$ at the point $t = 0$ has a singularity of infinity order.

Consider now problem (1_0), (2). Theorems 3 and 4 below concern the cases where the function τ satisfies, respectively, the inequalities

$$\tau(t) \le t \ \text{ for } \ 0 < t < a, \tag{10}$$

or

$$\tau(t) \ge t \ \text{ for } \ 0 < t < a. \tag{11}$$

Theorem 3. *Let conditions* (3), (4) *hold, or let condition* (3) *be violated and instead of* (4) *inequality* (4′) *be satisfied. Let, moreover, the function* τ *satisfy inequality* (10), *and the function* q *satisfy the condition*

$$\int_0^a q_n^{-\frac{\mu}{1+\mu}}(\tau(s))q(s)ds < +\infty.$$

Then problem $(1_0), (2)$ *has at least one solution.*

Corollary 1. *Let conditions* (3), (4) *hold, or let condition* (3) *be violated and instead of* (4) *inequality* (4′) *be satisfied. Let, moreover,*

$$\tau(t) = a^{1-\alpha}t^{\alpha}, \quad 0 < \liminf_{t\to 0} \big(t^{-\gamma}q(t)\big) \le \limsup_{t\to 0} \big(t^{-\gamma}q(t)\big), \tag{12}$$

where α *and* γ *are constants, satisfying the inequality*

$$\alpha \ge 1, \ \ \gamma > \mu|\alpha n + \alpha\gamma - \gamma - 1|. \tag{13}$$

Then problem $(1_0), (2)$ *has at least one solution.*

Theorem 4. *Let conditions* (3), (4) *hold, or let condition* (3) *be violated and instead of* (4) *inequality* (4′) *be satisfied. Let, moreover,* τ *be a nondecreasing function, satisfying inequality* (11), *and the function* q *be such that*

$$\int_0^a q_n^{-\frac{\mu}{1+\mu}}(s)q(s)ds < +\infty.$$

Then problem $(1_0), (2)$ *has at least one solution.*

Corollary 2. *Let conditions* (3), (4) *hold, or let condition* (3) *be violated and instead of* (4) *inequality* (4′) *be satisfied. Let, moreover, the functions* τ *and* q *satisfy conditions* (12), *where* α *and* γ *are constants such that*

$$0 < \alpha < 1, \ \ \gamma > \mu(n-1). \tag{14}$$

Then problem $(1_0), (2)$ *has at least one solution.*

As an example, consider the differential equation

$$u^{(n)}(t) = p_0(t)\exp\Big(\frac{\lambda(n-1)t}{a}\Big)u^{\lambda}\Big(a\exp\Big(\frac{-t}{a}\Big)\Big) + q_0(t)t^{\gamma}u^{-\mu}(a^{1-\alpha}t^{\alpha}). \tag{15}$$

Here p_0 and $q_0 :]0, a[\to [0, +\infty[$ are arbitrary integrable functions, satisfying the inequalities

$$\liminf_{t\to 0} p_0(t) > 0, \ \ 0 < \liminf_{t\to 0} q_0(t) \le \limsup_{t\to 0} q_0(t) < +\infty,$$

$\lambda \in]0,1]$, $\mu \in]0,+\infty[$, and α and γ are constants, satisfying either inequalities (13) or inequalities (14).

According to Corollaries 1 and 2, problem (15), (2) has at least one solution. On the other hand, it is evident that if $n > 1$, then Eq. (15) in the time variable has a singularity of infinite order at the point $t = 0$.

Finally, note that the question on the uniqueness of a solution of problem $(1_0), (2)$ (and, in particular, of problem (15), (2)) remains open.

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