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Oscillatory properties of solutions of higher order nonlinear functional differential equations

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We investigate oscillatory properties of solutions of the functional differential equation

$$u^{(n)}(t)=f(u)(t) \tag{1}$$

and its particular cases

$$u^{(n)}(t)=g\big(t,u(\tau_1(t)),\dots,u(\tau_m(t))\big), \tag{2}$$

$$u^{(n)}(t)=\sum_{k=1}^{m} g_k(t)\ln\big(1+|u(\tau_k(t))|\big)\operatorname{sgn}(u(\tau_k(t))). \tag{3}$$

Here, f is an operator acting from the space $C([a,+\infty[)$ to the space $L_{loc}(\mathbb{R}_+)$, $a\leq 0$, $g:\mathbb{R}_+\times\mathbb{R}^m\to\mathbb{R}$ is a function satisfying the local Carathéodory conditions,

$$g_k\in L_{loc}(\mathbb{R}_+)\ \ (k=1,\dots,m),$$

and $\tau_k:\mathbb{R}_+\to\mathbb{R}$ $(k=1,\dots,m)$ are continuous functions such that

$$\tau_k(t)\leq t\ \text{ for }\ t\in\mathbb{R}_+,\quad \lim_{t\to+\infty}\tau_k(t)=+\infty\ \ (k=1,\dots,m). \tag{4}$$

We use the following notation and definitions.
$p_0(t)\equiv 1$, $p_k(t)\equiv t^k$ $(k=1,2,\dots)$.
If k is a natural number, then $\mathcal{N}_k^0$ is the set of those $i\in\{1,\dots,k\}$ for which $i+k$ is even.
$\mathbb{R}$ is the set of real numbers, $\mathbb{R}_+=[0,+\infty[$.
$C([a,+\infty[)$ is the space of continuous functions $u:[a,+\infty[\,\to\mathbb{R}$.
If $u\in C([a,+\infty[)$ and $a_0\geq a$, then

$$\chi(a_0;u)(t)=\begin{cases} u(t) & \text{for }\ t\geq a_0,\\ u(a_0) & \text{for }\ t<a_0,\end{cases}$$

$$\nu(u)(t)=\max\big\{|u(s)|:\ a\leq s\leq t\big\}\ \text{ for }\ t>a.$$

$L_{loc}(\mathbb{R}_+)$ is the space of functions $v : \mathbb{R}_+ \to \mathbb{R}$, Lebesgue integrable on every finite interval contained in $\mathbb{R}_+$.

An operator $f_0 : C([a,+\infty[) \to L_{loc}(\mathbb{R}_+)$ is said to be a **Volterra operator** if for any $t > a$ and $u_i : C([a,+\infty[)$, satisfying the condition

$$u_1(s) = u_2(s) \ \text{ for } \ s \in [0,t],$$

we have

$$f_0(u_1)(s) = f_0(u_2)(s) \ \text{ for almost all } \ s \in [0,t].$$

An operator $f_0 : C([a,+\infty[) \to L_{loc}(\mathbb{R}_+)$ is said to be **continuous** if for any $u \in C([a,+\infty[)$ and any sequence $u_k \in C([a,+\infty[)$ $(k = 1,2,\dots)$, satisfying the condition

$$\lim_{k\to+\infty} v(u_k - u)(t) = 0 \ \text{ for } \ t \geq a,$$

the equality

$$\lim_{k\to+\infty} \int_0^t \Big| f_0(u_k)(s) - f_0(u)(s) \Big| \, ds = 0 \ \text{ for } \ t > 0$$

holds.

Everywhere below, it is assumed that $f : C([a,+\infty[) \to L_{loc}(\mathbb{R}_+)$ is a continuous Volterra operator.

Let $t_0 \geq 0$. An $(n-1)$-times continuously differentiable function $u : [t_0,+\infty[\to \mathbb{R}$ is said to be a **solution of equation** (1) if $u^{(n-1)}$ is absolutely continuous on every finite interval contained in $[t_0,+\infty[$, and there exists a continuous function $u_0 : [a,t_0] \to \mathbb{R}$ such that almost everywhere on $[t_0,+\infty[$ equality (1) is fulfilled, where

$$u(t) = u_0(t) \ \text{ for } \ a \leq t \leq t_0.$$

A solution u of equation (1) defined on some interval $[t_0,+\infty[\subset \mathbb{R}_+$ is said to be **proper** if it does not identically equal to zero in any neighbourhood of $+\infty$.

A proper solution $u : [t_0,+\infty[\subset \mathbb{R}$ is said to be **oscillatory** if it changes sign in any neighbourhood of $+\infty$, and it is said to be a **Kneser solution** if on some interval $[t_1,+\infty[\subset [t_0,+\infty[$ it satisfies the inequalities

$$(-1)^i u^{(i)}(t) u(t) > 0 \ \ (i = 1,\dots,n-1).$$

Equation (1) has the **property** A_0 if every its proper solution for n even is oscillatory and for n odd either is oscillatory or is a Kneser solution.

Equation (1) has the **property** B_0 if every its proper solution for n even is either oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty} |u^{(n-2)}(t)| = +\infty, \tag{5}$$

and for n odd either is oscillatory or satisfies condition (5).

Unlike the properties A and B whose notions in the oscillation theory have been introduced by V. A. Kondrat'ev [11] and I. Kiguradze [5] the properties A_0 and B_0 do not assume that Kneser solutions of the equation under consideration are vanishing at infinity, and the unbounded solutions satisfy the harder than (5) condition

$$\lim_{t\to+\infty} |u^{(n-1)}(t)| = +\infty.$$

We have found integral conditions under which equation (1) has, respectively, the properties A_0 and B_0. In contrast to the well-known earlier results (see, e.g., [1, 2, 3, 4, 6, 7, 8, 9, 10, 12, 13, 14] and the references therein), the proven by us general oscillation theorems yield the necessary and sufficient conditions for equation (3) to have the properties A_0 and B_0 in the case where g_k $(k = 1, \dots, m)$ are of the constant sign functions of the same sign.

We investigate oscillatory properties of equation (1) in the case where the operator f is monotone, or more precisely, when f satisfies one of the following two conditions:

(M_-): $f(0)(t) \equiv 0$ and for any numbers $t_1 \geq 0$, $t_2 > t_1$, $t_0 \in [t_1, t_2]$, and functions $u_i \in C(\mathbb{R}_+)$ $(i = 1, 2)$, satisfying the condition

$$u_1(t_0) \neq 0, \quad u_i(t)u_1(t_0) \geq 0 \;\; (i = 1, 2), \quad u_1(t) \leq u_2(t) \;\text{ for }\; t_1 \leq t \leq t_2, \tag{6}$$

almost everywhere on $[t_1, t_2]$ the inequality

$$f(\chi(t_1; u_1))(t) \geq f(\chi(t_1; u_2))(t)$$

is fulfilled.

(M_+): $f(0)(t) \equiv 0$ and for any numbers $t_1 \geq 0$, $t_2 > t_1$, $t_0 \in [t_1, t_2]$, and functions $u_i \in C(\mathbb{R}_+)$ $(i = 1, 2)$, satisfying condition (6), almost everywhere on $[t_1, t_2]$ the inequality

$$f(\chi_1(t_1; u_1))(t) \leq f(\chi(t_1; u_2))(t)$$

is fulfilled.

Theorem 1. *If the operator f satisfies condition (M_-) and*

$$\int\limits_0^{+\infty} t^{n-i-1} |f(xp_{i-1})(t)| \, dt = +\infty \;\; \text{for} \;\; x \neq 0, \quad i \in \mathcal{N}_{n-1}^0, \tag{7}$$

then equation (1) *has the property A_0.*

Theorem 2. *If the operator f satisfies condition (M_+) and*

$$\int\limits_0^{+\infty} t^{n-i-1} |f(xp_{i-1})(t)| \, dt = +\infty \;\; \text{for} \;\; x \neq 0, \quad i \in \mathcal{N}_{n-1}^0, \tag{8}$$

then equation (1) *has the property B_0.*

Conditions (7) and (8) in Theorems 1 and 2 are unimprovable. In particular, the following theorems hold.

Theorem 3. *Let the operator f satisfy condition (M_{-}) and for any $x \neq 0$ there exist numbers $t_x \geq 0$ and $\delta(x) > 0$ such that*

$$t^{n-i-1}|f(xp_{i-1})(t)| \geq \delta(x)|f(xp_{n-1})(t)| \ \text{ for } \ t \geq t_x, \ \ i \in \mathcal{N}^0_{n-1}.$$

Then for equation (1) to have the property A_0, it is necessary and sufficient that equalities (7) be satisfied.

Theorem 4. *Let $n \geq 3$, the operator f satisfy condition (M_{-}), and for any $x \neq 0$ there exist numbers $t_x \geq 0$ and $\delta(x) > 0$ such that*

$$t^{n-i-2}|f(xp_{i-1})(t)| \geq \delta(x)|f(xp_{n-2})(t)| \ \text{ for } \ t \geq t_x, \ \ i \in \mathcal{N}^0_{n-2}.$$

Then for equation (1) to have the property B_0, it is necessary and sufficient that equalities (8) be fulfilled.

Below everywhere, when discussing equations (2) and (3), we assume that the functions τ_i $(i = 1, \dots, n)$ satisfy condition (4).

We investigate equation (2) in the case where the function g satisfies one of the following two conditions:

$$g(t, 0, \dots, 0) = 0, \quad g(t, x_1, \dots, x_m) \geq g(t, y_1, \dots, y_m) \ \text{ for } \ t > 0, \ \ x_i x_1 > 0, \ \ y_i x_1 > 0, \ \ x_i \leq y_i \ \ (i = 1, \dots, m) \tag{9}$$

and

$$g(t, 0, \dots, 0) = 0, \quad g(t, x_1, \dots, x_m) \leq g(t, y_1, \dots, y_m) \ \text{ for } \ t > 0, \ \ x_i x_1 > 0, \ \ y_i x_1 > 0, \ \ x_i \leq y_i \ \ (i = 1, \dots, m). \tag{10}$$

Theorems 1 and 3 imply the following corollaries.

Corollary 1. *If the function g satisfies condition (9) and*

$$\int\limits_0^{+\infty} t^{n-i-1} \Big| g\big(t, x|\tau_1(t)|^{i-1}, \dots, x|\tau_m(t)|^{i-1}\big) \Big| \, dt = +\infty \ \text{ for } \ x \neq 0, \ \ i \in \mathcal{N}^0_{n-1}, \tag{11}$$

then equation (2) has the property A_0.

Corollary 2. *Let the function g satisfy condition (9) and for any $x \neq 0$ there exist numbers $t_x > 0$ and $\delta(x) > 0$ such that*

$$t^{n-i-1} \Big| g\big(t, x|\tau_1(t)|^{i-1}, \dots, x|\tau_m(t)|^{i-1}\big) \Big| \geq \delta(x) \Big| g\big(t, x|\tau_1(t)|^{n-1}, \dots, x|\tau_m(t)|^{n-1}\big) \Big| \ \text{ for } \ t > t_x, \ \ i \in \mathcal{N}^0_{n-1}.$$

Then for equation (2) to have the property A_0, it is necessary and sufficient that equalities (11) be fulfilled.

Theorems 3 and 4 for equation (2) take the following forms.

Corollary 3. *If the function g satisfies condition (10) and*

$$\int_0^{+\infty} t^{n-i-1}\Big|g\big(t, x|\tau_1(t)|^{i-1},\dots,x|\tau_m(t)|^{i-1}\big)\Big|\,dt = +\infty \ \ \text{for} \ \ x \neq 0, \ \ i \in \mathcal{N}^0_{n-2}, \tag{12}$$

then equation (2) has the property B_0.

Corollary 4. *Let $n \geq 3$, the function g satisfy condition (10), and for any $x \neq 0$ there exist numbers $t_x > 0$ and $\delta(x) \neq 0$ such that*

$$t^{n-i-2}\Big|g\big(t, x|\tau_1(t)|^{i-1},\dots,x|\tau_m(t)|^{i-1}\big)\Big| \\ \geq \delta(x)\Big|g\big(t, x|\tau_1(t)|^{n-2},\dots,x|\tau_m(t)|^{n-2}\big)\Big| \ \ \text{for} \ \ t > t_x, \ \ i \in \mathcal{N}^0_{n-2}.$$

Then for equation (2) to have the property B_0, it is necessary and sufficient that equalities (12) be fulfilled.

Finally, let us consider equation (3). Corollaries 2 and 3 result in the following corollaries.

Corollary 5. *If $n > 2$ and*

$$g_k(t) \leq 0 \ \ \text{for} \ \ t > 0 \ \ (k = 1,\dots,m),$$

then for equation (3) to have the property A_0, it is necessary and sufficient that the equality

$$\int_0^{+\infty}\Big(\sum_{k=1}^m g_k(t)\ln(1+|\tau_k(t)|)\Big)\,dt = -\infty$$

be fulfilled.

Corollary 6. *If $n > 3$ and*

$$g_k(t) \geq 0 \ \ \text{for} \ \ t > 0 \ \ (k = 1,\dots,m),$$

then for equation (3) to have the property B_0, it is necessary and sufficient that the equality

$$\int_0^{+\infty} t\Big(\sum_{k=1}^m g_k(t)\ln(1+|\tau_k(t)|)\Big)\,dt = +\infty$$

be fulfilled.

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