On the Cauchy–Nicoletti Weighted Problem for Nonlinear Singular Functional Differential Systems

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Let $-\infty < a < b < +\infty$, $t_i \in [a,b]$ $(i=1,\dots,n)$, $\varphi = (\varphi_i)_{i=1}^n : [a,b] \to R^n$ be a continuous vector function such that

$$\varphi_i(t_i) = 0, \quad \varphi_i(t) > 0 \ \text{ for } \ t \neq t_i \ \ (i=1,\dots,n),$$

and let $C_\varphi([a,b];R^n)$ be the space of continuous vector functions $x = (x_i)_{i=1}^n : [a,b] \to R^n$, satisfying the condition

$$\|x\|_{C_\varphi} = \sum_{i=1}^n \|x_i\|_{C_{\varphi_i}} < +\infty,$$

where

$$\|x_i\|_{C_{\varphi_i}} = \sup\left\{ \frac{|x_i(t)|}{\varphi_i(t)} : \ a \le t \le b, \ t \neq t_i \right\} \ \ (i=1,\dots,n).$$

Let, moreover, $I_k = [a,b] \setminus \{t_k\}$ $(k=1,\dots,n)$, and $L_{loc}(I_k;R)$ be the space of functions $u : I_k \to R$, Lebesgue integrable on every closed interval, contained in I_k.

We consider the nonlinear functional differential system

$$\frac{dx_i(t)}{dt} = f_i(x_1,\dots,x_n)(t) \ \ (i=1,\dots,n) \tag{1}$$

with the weighted boundary conditions

$$\limsup_{t\to t_i} \frac{|x_i(t)|}{\varphi_i(t)} < +\infty \ \ (i=1,\dots,n). \tag{2}$$

Here the operators $f_k : C_\varphi([a,b];R^n) \to L_{loc}(I_k;R)$ $(k=1,\dots,n)$ are such that:

(i) for any $\rho > 0$, the condition

$$f_k^*(\varphi,\rho) \in L_{loc}(I_k;R) \ \ (k=1,\dots,n)$$

is satisfied, where

$$f_k^*(\varphi,\rho)(t) = \sup\left\{ |f_k(x_1,\dots,x_n)(t)| : \ \|(x_i)_{i=1}^n\|_{C_\varphi} \le \rho \right\};$$

(ii) for any $\rho > 0$, $k \in \{1,\dots,n\}$, $[a_0,b_0] \subset I_k$, and uniformly converging sequence of continuous vector functions $(x_{im})_{i=1}^n : [a,b] \to R^n$ $(m=1,2,\dots)$, satisfying the conditions

$$\|(x_{im})_{i=1}^n\|_{C_\varphi} \le \rho \ \ (m=1,2,\dots), \tag{3}$$
$$\lim_{m\to\infty} (x_{im}(t))_{i=1}^n = (x_i(t))_{i=1}^n \ \ \text{for} \ \ t\in[a,b],$$

we have

$$\lim_{m\to\infty} \int_{a_0}^t f_k(x_{1m},\dots,x_{nm})(s)\,ds$$
$$= \int_{a_0}^t f_k(x_1,\dots,x_n)(s)\,ds \ \ \text{uniformly on} \ \ [a_0,b_0].$$

We are interested in the case where system (1) is singular, i.e. the case where

$$\int_a^b \bigg(\sum_{k=1}^n f_k^*(\varphi,\rho)(t)\bigg)\,dt = +\infty \ \ \text{for any} \ \ \rho>0.$$

From (2) we have the conditions

$$x_i(t_i) = 0 \ \ (i=1,\dots,n). \tag{4}$$

Problem (1), (2) we call the Cauchy–Nicoletti weighted problem since problem (1), (4) is known as the Cauchy–Nicoletti problem (see, [1, 2, 4, 6, 7]).

By I. Kiguradze [2, 3], the optimal sufficient conditions are obtained for the solvability and unique solvability of the Cauchy–Nicoletti weighted problem for systems of ordinary differential equations with singularities with respect to a time variable at the points bearing the boundary data. In the present work, these results are generalized for functional differential systems. Precisely, unimprovable in a certain sense conditions are established guaranteeing, respectively, the solvability and unsolvability of problem (1), (2) and the stability of its solution with respect to small perturbations of the right-hand sides of system (1).

Everywhere below, in addition to the above-introduced notation, we will also use the following notation.

$X = (x_{ik})_{i,k=1}^n$ is the $n\times n$ matrix with components x_{ik} $(i,k=1,\dots,n)$.

$r(X)$ is the spectral radius of an $n\times n$ matrix X.

$[x]_+$ is the positive part of a real number x, i.e., $[x]_+ = (x+|x|)/2$.

For any $i\in\{1,\dots,n\}$, $\delta>0$, and $\lambda\in[0,1]$, we set

$$\chi_i(t,\delta,\lambda) = \begin{cases} 0 & \text{for } \ t\in[t_i-\delta,t_i+\delta], \\ \lambda & \text{for } \ t\notin[t_i-\delta,t_i+\delta], \end{cases}$$

and along with (1) we consider the auxiliary differential system

$$\frac{dx_i(t)}{dt} = \chi_i(t,\delta,\lambda) f_i(x_1,\dots,x_n)(t) \ \ (i=1,\dots,n). \tag{5}$$

On the basis of Corollary 2 from [5], the following theorem can be proved.

Theorem 1 (The principle of a priori boundedness)**.** *Let there exist a positive number ρ such that for any $\lambda \in [0,1]$ and $\delta > 0$ every solution of problem* (5), (4) *admits estimate* (3). *Then problem* (1), (2) *has at least one solution.*

This theorem enables us to establish efficient sufficient conditions for the solvability of problem (1), (2).

In particular, the following theorem is valid.

Theorem 2. *Let there exist nonnegative numbers p_{ik} $(i,k = 1,\dots,n)$ and a function $q : [0,+\infty[\to [0,+\infty[$ such that*

$$r(P) < 1, \quad where \ \ P = (p_{ik})_{i,k=1}^n, \tag{6}$$

$$\lim_{s\to+\infty} \frac{q(s)}{s} = 0,$$

and for any $(x_i)_{i=1}^n \in C_\varphi([a,b];R^n)$ the inequalities

$$\left| \int_{t_i}^t \big[f_i(x_1,\dots,x_n)(s)\operatorname{sgn}\big((s-t_i)x_i(s)\big)\big]_+ \, ds \right| \\ \le \varphi_i(t)\left(\sum_{k=1}^n p_{ik}\|x_k\|_{C_{\varphi_k}} + q\left(\sum_{k=1}^n \|x_k\|_{C_{\varphi_k}}\right)\right) \quad (i=1,\dots,n)$$

are satisfied in the interval $[a,b]$. Then problem (1), (2) *has at least one solution.*

We consider now the perturbed problem

$$\frac{dy_i(t)}{dt} = f_i(y_1,\dots,y_n)(t) + h_i(t) \quad (i=1,\dots,n), \tag{7}$$

$$\limsup_{t\to t_i} \frac{|y_i(t)|}{\varphi_i(t)} < +\infty \quad (i=1,\dots,n), \tag{8}$$

and we introduce the following

Definition. Problem (1), (2) is said to be well-posed if:

(i) it has a unique solution $(x_i)_{i=1}^n$;

(ii) there exists a positive constant ρ such that for any integrable functions $h_i :]a,b[\to R$ $(i=1,\dots,n)$, satisfying the conditions

$$\nu_k(h_k) = \sup\left\{ \frac{1}{\varphi_k(t)} \left| \int_{t_k}^t |h_k(s)|\,ds \right| : \ a \le t \le b, \ t \ne t_k \right\} < +\infty \quad (k=1,\dots,n),$$

problem (7), (8) has at least one solution and every such solution admits the estimate

$$\left\| (y_i - x_i)_{i=1}^n \right\|_{C_\varphi} \le \rho \sum_{k=1}^n \nu_k(h_k).$$

Theorem 3. *Let there exist nonnegative numbers p_{ik} $(i,k=1,\dots,n)$, satisfying condition* (6)*, such that for any $(x_i)_{i=1}^n \in C_\varphi([a,b];R^n)$, in the interval $[a,b]$ the inequalities*

$$\bigg|\int_{t_i}^{t} \big[f_i(x_1,\dots,x_n)(s)\operatorname{sgn}((s-t_i)x_i(s))\big]_+\,ds\bigg| \le \varphi_i(t)\sum_{k=1}^{n} p_{ik}\|x_k\|_{C_{\varphi_k}} \quad (i=1,\dots,n)$$

are fulfilled. Then problem (1)*,* (2) *is well-posed.*

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