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Two-Point Boundary Value Problems for Higher Order Singular Differential Equations

Ivan Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia

ivane.kiguradze@tsu.ge

In an open finite interval $]a,b[$, we consider the differential equation

$$u^{(2m)} = f(t,u) \tag{1}$$

with the boundary conditions of one of the following three types

$$u^{(i-1)}(a) = 0, \quad u^{(i-1)}(b) = 0 \ \ (i = 1,\dots,m), \tag{2}$$

$$u^{(2i-2)}(a) = 0, \quad u^{(2i-2)}(b) = 0 \ \ (i = 1,\dots,m), \tag{3}$$

$$u(a) = 0, \ \ u(b) = 0, \quad u^{(i)}(a) = u^{(i)}(b) \ \ (i = 1,\dots,2m-2). \tag{4}$$

Here, $f :]a,b[\times \mathbb{R} \to \mathbb{R}$ is a function satisfying the local Carathéodory conditions, i.e., f is measurable in the first argument, continuous in the second argumant and for arbitrarily small $\varepsilon > 0$ and arbitrary $x > 0$ the condition

$$\int\limits_{a+\varepsilon}^{b-\varepsilon} f^*(t,x)\,dt < +\infty$$

is fulfilled, where

$$f^*(t,x) = \max\big\{|f(t,y)| : \ |y| \le x\big\}.$$

Of our interest is the case when equation (1) is singular with respect to the time variable, i.e., the case, where

$$\int\limits_a^b f^*(t,x)\,dt = +\infty.$$

We have established new sufficient conditions of solvability and unique solvability of problems (1), (2); (1), (3) and (1), (4) covering the case when equation (1) is super-linear and its right-hand side has with respect to the time variable singularities of arbitrary order at the ends of the segment $]a,b[$.

In the existence theorems below it is assumed that the function f in the domain $]a,b[\times\mathbb{R}$ satisfies the one-sided restriction

$$(-1)^m f(t,x)\operatorname{sgn}(x) \le p(t)|x| + q(t), \tag{5}$$

where $p,q :]a,b[\to [0,+\infty[$ are the functions, Lebesgue integrable on every closed interval contained in $]a,b[$. The uniqueness theorems are concerned with the cases when the function f in the domain $]a,b[\times\mathbb{R}$ instead of (5) satisfies the one-sided Lipschitz condition

$$(-1)^m\big[f(t,x)-f(t,y)\big]\operatorname{sgn}(x-y) \le p(t)|x-y|. \tag{6}$$

We use the following notation.

$$\delta_m(t) = \frac{(t-a)^{2m-1}(b-t)^{2m-1}}{(2m-1)[(m-1)!]^2\,[(t-a)^{2m-1}+(b-t)^{2m-1}]}\,.$$

$C^k([a,b])$ is the space of k times continuously differentiable functions $u : [a,b]\to\mathbb{R}$.

$\widetilde{C}^k_{loc}(]a,b[)$ is the space of functions $u :]a,b[\to\mathbb{R}$ which are absolutely continuous together with $u',\dots,u^{(k)}$ on every closed interval contained in $]a,b[$.

$\widetilde{C}^{2m-1,m}(]a,b[)$ is the space of functions $u\in C^{m-1}([a,b])\cap\widetilde{C}^{2m-1}_{loc}(]a,b[)$ such that

$$\int_a^b |u^{(m)}(t)|^2\,dt < +\infty.$$

Everywhere below, when the question concerns problem (1), (3) or (1), (4), it is assumed that $m>2$.

A solution of problem (1), (2) is sought in the space $\widetilde{C}^{2m-1,m}(]a,b[)$, and solutions of problems (1), (3) and (1), (4) – in the space $\widetilde{C}^{2m-1}_{loc}(]a,b[)\cap C^{2m-2}([a,b])$.

Theorem 1. *If along with* (5) *the conditions*

$$\int_a^b \delta_m(t)p(t)\,dt < 1, \tag{7}$$

$$\int_a^b \big[(t-a)(b-t)\big]^{m-1/2}q(t)\,dt < +\infty \tag{8}$$

are fulfilled, then problem (1), (2) *has at least one solution in the space* $\widetilde{C}^{2m-1,m}(]a,b[)$.

Theorem 2. *If along with* (6) *conditions* (7) *and* (8), *where* $q(t)\equiv|f(t,0)|$, *are fulfilled, then problem* (1), (2) *has one and only one solution.*

Remark 1. If $m>1$, then for inequality (7) to be fulfilled it is sufficient that the inequality

$$\int_a^b \big[(t-a)(b-t)\big]^{2m-1}p(t)\,dt \le 4^{1-m}(2m-1)[(m-1)!]^2(b-a)^{2m-1}$$

or the inequality

$$\int_a^b p(t)\,dt \le \frac{4^m(2m-1)[(m-1)!]^2}{(b-a)^{2m-1}}$$

be satisfied.

Theorem 3. *Let condition* (5) *be fulfilled and there exist a number $\alpha \ge 1$ such that*

$$\int_a^b (t-a)^2(b-t)^2 p^\alpha(t)\,dt < 3(b-a)\Big(\frac{\pi}{b-a}\Big)^{2m\alpha-4}. \tag{9}$$

If, moreover,

$$\int_a^b (t-a)(b-t)\big[f^*\big(t,(t-a)(b-t)x\big) + q(t)\big]\,dt < +\infty \ \ \text{for} \ \ x > 0, \tag{10}$$

then problem (1), (3) *has at least one solution.*

Theorem 4. *If along with* (6) *conditions* (9), (10)*, where $\alpha \ge 1$ and $q(t) \equiv |f(t,0)|$, are fulfilled, then problem* (1), (3) *has one and only one solution.*

Corollaries of Theorems 3 and 4 deal with the case where $p(t) \equiv p = const \ge 0$, i.e., the case where instead of (5) and (6), respectively, the conditions

$$(-1)^m f(t,x)\,\mathrm{sgn}(x) \le p|x| + q(t), \tag{11}$$

$$(-1)^m\big[f(t,x) - f(t,y)\big]\,\mathrm{sgn}(x-y) \le p|x-y| \tag{12}$$

are fulfilled.

Corollary 1. *If along with* (11) *condition* (10) *is fulfilled and*

$$p < \Big(\frac{\pi}{b-a}\Big)^{2m}, \tag{13}$$

then problem (1), (3) *has at least one solution.*

Corollary 2. *If along with* (12) *conditions* (10) *and* (13)*, where $q(t) \equiv |f(t,0)|$, are fulfilled, then problem* (1), (3) *has one and only one solution.*

Remark 2. Inequality (13) in Corollaries 1 and 2 is unimprovable and it cannot be replaced by the nonstrict inequality

$$p \le \Big(\frac{\pi}{b-a}\Big)^{2m}. \tag{14}$$

Indeed, if

$$f(t,x) = (-1)^m\Big(\frac{\pi}{b-a}\Big)^{2m} x + 1,$$

then problem (1), (3) has no solution, although, in this case all the conditions of Corollaries 1 and 2 are fulfilled, except (13), instead of which condition (14) is satisfied.

Theorem 5. *Let conditions* (5), (10) *be fulfilled and there exist a number* $\alpha \geq 1$ *such that*

$$\int_a^b (t-a)^2(b-t)^2 p^\alpha(t)\,dt < 6(b-a)2^{-\alpha}\Big(\frac{2\pi}{b-a}\Big)^{2m\alpha-4}. \tag{15}$$

Then problem (1), (4) *has at least one solution.*

Theorem 6. *If conditions* (6), (10) *and* (15), *where* $\alpha \geq 1$ *and* $q(t) \equiv |f(t,0)|$, *are fulfilled, then the problem* (1), (4) *has one and only one solution.*

Corollary 3. *If along with* (11) *condition* (10) *is fulfilled and*

$$p < \frac{1}{2}\Big(\frac{2\pi}{b-a}\Big)^{2m}, \tag{16}$$

then problem (1), (4) *has at least one solution.*

Corollary 4. *If along with* (12) *conditions* (10) *and* (16), *where* $q(t) \equiv |f(t,0)|$, *are fulfilled, then problem* (1), (4) *has one and only one solution.*

As an example, let us consider the differential equation

$$u^{(2m)} = (-1)^m \big[p(t)u - g(t)|u|^\lambda \operatorname{sgn} u\big] + h(t), \tag{17}$$

where λ is a positive constant, $p :]a,b[\to [0,+\infty[$, $g :]a,b[\to [0,+\infty[$ and $h :]a,b[\to \mathbb{R}$ are the functions, Lebesgue integrable on every closed interval contained in $]a,b[$.

If the function p satisfies condition (7) and

$$\int_a^b t^{m-1/2}|h(t)|\,dt < +\infty,$$

then, according to Theorem 2, problem (17), (2) has one and only one solution.

In this case, the function g may have singularities of arbitrary type at the points a and b.

If, however, the function p satisfies condition (9) (condition (15)) and

$$\int_a^b \Big[\big[(t-a)(b-t)\big]^{\lambda+1} g(t) + (t-a)(b-t)|h(t)|\Big]\,dt < +\infty,$$

then by Theorem 4 (Theorem 6), problem (17), (3) (problem (17), (4)) has one and only one solution.