On Oscillation of Solutions to Second-Order Emden–Fowler type Differential Equations with Positive Potential

K. Dulina, T. Korchemkina

Lomonosov Moscow State University,
sun-ksi@mail.ru, krtaalex@gmail.com

1 Introduction

Consider the second-order Emden–Fowler type differential equation

$$y'' + p(x,\, y,\, y')\, |y|^k \operatorname{sgn} y = 0, \quad k > 0,\, k \neq 1, \tag{1}$$

where the function $p(x,\, u,\, v)$ defined on $\mathbb{R} \times \mathbb{R}^2$ is positive, continuous in x, Lipschitz continuous in u, v.

Asymptotic behavior of all solutions to equation (1) in the case $p = p(x)$ was described by I.T. Kiguradze and T.A. Chanturia (see [1]). Properties of oscillating solutions to third- and fourth-order similar differential equations are described in [2, 3]. The oscillating criteria for solutions to high-order Emden–Fowler type differential equations is given in [4]. Results on asymptotic classification of maximally extended solutions to third- and fourth-order differential equations with negative potential for $k > 0$, $k \neq 1$ are given by I.V. Astashova (see [3, 5, 6, 7]). Asymptotic classification of solutions to equation (1) with regular ($k > 1$) and singular ($0 < k < 1$) nonlinearity for the bounded negative function $p(x,\, u,\, v)$ is contained in [8]. Asymptotic behavior of maximally extended solutions for the unbounded negative function $p(x,\, u,\, v)$ is investigated in [9, 10].

Further suppose the function $p(x,\, u,\, v)$ satisfies inequalities

$$0 < m \le p(x,\, u,\, v) \le M < +\infty. \tag{2}$$

2 Behavior of maximally extended solutions

The following statements describe the behavior of solutions to equation (1).

Theorem 1 *All nontrivial maximally extended solutions to equation* (1) *and their first derivatives are oscillating at increasing and decreasing argument. Moreover, zeroes x_j of solutions and zeroes x'_j of their first derivatives alternate, i. e.*

$$\cdots < x_{j-1} < x'_j < x_j < x'_{j+1} < \ldots, \quad j \in \mathbb{Z}.$$

Lemma 1 *Let $y(x)$ be a nontrivial maximally extended solution to equation* (1). *Then for any $j \in \mathbb{Z}$ the following inequalities hold:* $-\sqrt{\frac{M}{m}} \le \frac{y'(x_{j+1})}{y'(x_j)} \le -\sqrt{\frac{m}{M}}$.

Lemma 2 *Let $y(x)$ be a nontrivial maximally extended solution to equation* (1). *Then for any $j \in \mathbb{Z}$ the following inequalities hold:* $-\left(\frac{M}{m}\right)^{\frac{2}{k+1}} \le \frac{y\left(x'_{j+1}\right)}{y\left(x'_j\right)} \le -\left(\frac{m}{M}\right)^{\frac{2}{k+1}}$.

Denote

$$M_j = \max_{x \in [x_j,\, x_{j+1}]} p\left(x,\, y(x),\, y'(x)\right), \quad m_j = \min_{x \in [x_j,\, x_{j+1}]} p\left(x,\, y(x),\, y'(x)\right), \quad j \in \mathbb{Z}.$$

Remark 1 *For any $j \in \mathbb{Z}$ the following inequalities hold:*

$$-\sqrt{\frac{M_j}{m_j}} \le \frac{y'\left(x_{j+1}\right)}{y'\left(x_j\right)} \le -\sqrt{\frac{m_j}{M_j}}, \quad -\left(\frac{M_j^2}{m_j\, m_{j-1}}\right)^{\frac{1}{k+1}} \le \frac{y\left(x'_{j+1}\right)}{y\left(x'_j\right)} \le -\left(\frac{m_j^2}{M_j\, M_{j-1}}\right)^{\frac{1}{k+1}}.$$

Note in the case $p(x,\, u,\, v) \equiv p_0 > 0$ all the nontrivial maximally extended solutions to equation (1) are periodical ones.

Theorem 2 *Let $y(x)$ be a nontrivial maximally extended solution to equation* (1). *Suppose the function $p(x,\, u,\, v)$ continuous in x, Lipschitz continuous in u, v and satisfying inequalities* (2). *Let the function $p(x,\, u,\, v)$ also tend to $p_+ > 0$ as $x \to +\infty$ and tend to $p_- > 0$ as $x \to -\infty$ uniformly in u, v.*

Then $y(x)$ is defined on the whole axis and the following relations hold as $j \to \pm\infty$:

1) $\frac{y'(x_{j+1})}{y'(x_j)} \to -1$,

2) $\frac{y(x'_{j+1})}{y(x'_j)} \to -1$.

References

[1] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differential equations.* Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[2] I. V. Astashova, On asymptotic behavior of alternating solutions to certain nonlinear differential equations of the third and forth order, *Reports of extended session of a seminar of the I.N. Vekua Institute of Applied Mathematics, Tbilisi*, 1988, no. 3(3), 9–12. [in Russian]

[3] I. V. Astashova. *Qualitative properties of solutions to quasilinear ordinary differential equations. In: Astashova I. V. (ed.) Qualitative properties of solutions to differential equations and related topics of spectral analysis: scientific edition*, UNITY-DANA Publ., Moscow, 2012, 22–290. [in Russian].

[4] I. V. Astashova, On existence of non-oscillatory solutions to quasi-linear differential equations *Georgian Mathematical Journal*, **14(2)** (2007), 223–238.

[5] I. V. Astashova, On quasi-periodic solutions to a higher-order emden-fowler type differential equation, *Boundary Value Problems*, **2014(174)** (2014), 1–8.

[6] I. V. Astashova, On asymptotic classification of solutions to nonlinear third- and fourth- order differential equations with power nonlinearity, *Vestnik MSTU. Ser. "Estestvennye nauki"*, **2** (2015), 2–25.

[7] I. V. Astashova, On asymptotic classification of solutions to fourth-order differential equations with singular power nonlinearity, *Mathematical Modeling and Analysis*, **21(4)** (2016), 502–521.

[8] K. M. Dulina and T. A. Korchemkina, Asymptotic classification of solutions to second-order Emden–Fowler type differential equations with negative potential, *Vestnik of Samara State University. Estestv. Ser.*, **128(6)** (2015), 50–56. [in Russian]

[9] K. M. Dulina, On asymptotic behavior of solutions to the second-order Emden–Fowler type differential equations with unbounded negative potential, *Functional differential equations*, **23(1-2)** (2016), 3–8.

[10] K. M. Dulina, On asymptotic behavior of solutions with infinite derivative for regular second-order Emden–Fowler type differential equations with negative potential, *The Bulletin of Udmurt University. Mathematics. Mechanics. Computer Science*, **26(2)** (2016), 207–214. [in Russian]