On I.T.Kiguradze's problem concerning power-law asymptotic behavior of blow-up solutions to Emden–Fowler type differential equations.

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1 Introduction

Consider the equation

$$y^{(n)} = P(x, y, y', \dots, y^{(n-1)})\,|y|^k \operatorname{sgn} y,\ \ n \ge 2,\ \ k \in \mathbb{R},\quad k > 1, \tag{1}$$

where the positive function P is continuous in x and Lipschitz continuous in the last n variables. Consider also a special case of (1), namely

$$y^{(n)} = p_0\,|y|^k \operatorname{sgn} y,\quad n \ge 2,\quad k \in \mathbb{R},\quad k > 1,\quad p_0 > 0. \tag{2}$$

Definition 1. A solution $y(x)$ of equation (1) is said to be *n-positive* if it is maximally extended in both directions and eventually satisfies the inequalities

$$y(x) > 0,\ y'(x) > 0, \dots,\ y^{(n-1)}(x) > 0.$$

Note that if the above inequalities are satisfied by a solution of (2) at some point x_0, then they are also satisfied at any point $x > x_0$ in the domain of the solution. Moreover, such a solution, if maximally extended, must be a so-called blow-up solution (having a vertical asymptote at the right endpoint of its domain).

Hereafter we use the notation

$$\alpha = \frac{n}{k-1}. \tag{3}$$

Immediate calculations show that equation (2) has n-positive solutions defined on $(-\infty,\ x^*)$ with arbitrary $x^* \in \mathbb{R}$ and having exact power-law behavior, namely

$$y(x) = C(x^* - x)^{-\alpha},\quad C = \left(\frac{\alpha(\alpha+1)\dots(\alpha+n-1)}{p_0}\right)^{\frac{1}{k-1}}. \tag{4}$$

I. T. Kiguradze [1, Problem 16.4] posed a question on the equivalence, as $x \to x^*$, of all positive blow-up solutions of (2) with the vertical asymptote $x = x^*$ to the solution defined by (4).

For $n = 1$ all n-positive solutions of (2) are defined by (4). For $n \in \{2, 3, 4\}$ it is known that any n-positive solution of (2) and even of more general equations (1) is asymptotically equivalent, near the right endpoint of its domain, to the solution defined by (4) with appropriate x^*:

$$y(x) = C(x^* - x)^{-\alpha}(1 + o(1)),\ \ x \to x^* - 0. \tag{5}$$

(See [1] for $n = 2$, and [2], [3], [4] for $n \in \{3, 4\}$). For equation (1) we mean by $p_0 = \text{const} > 0$ in (4) the limit of $P(x,\, y_0,\, \dots, y_{n-1})$ as $x \to x^* - 0$, $y_0 \to \infty, \dots, y_{n-1} \to \infty$.

For equation (1) with some additional assumptions on the function P the existence of solution with power-law asymptotic behavior (5) is proved. For $5 \le n \le 11$, the existence of an $(n-1)$-parametrical family of such solutions is obtained (see [4]).

The natural hypothesis generalizing this statement for all $n > 4$ appears to be wrong even for equation (2) (see [5] for sufficiently large n and [6] for $n \in \{12, 13, 14\}$).

2 Existence of positive solutions with non power-law asymptotic behavior

For equation (2) it was proved [5] that for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (2) has a solution of the form

$$y = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha} \; h(\log{(x^* - x)}),$$

where α is defined by (3) and h is a positive periodic non-constant function on $\mathbf{R}$.

As for the question of how large should be n for the existence of that type of positive solutions, the following partial answer is given [6].

Theorem 1. *If* $12 \le n \le 14$, *then there exists* $k > 1$ *such that equation* (2) *has a solution* $y(x)$ *satisfying*

$$y^{(j)}(x) = p_0^{-\frac{1}{k-1}} (x^* - x)^{-\alpha - j} \, h_j(\, \log(x^* - x) \,), \qquad j = 0, 1, \ldots, n-1,$$

where α *is defined by* (3) *and* h_j *are periodic positive non-constant functions on* $\mathbf{R}$.

Remark 1. Computer calculations give approximate values of α providing the existence of the above-type solutions. They are, with the corresponding values of k, as follows:
if $n = 12$, then $\alpha \approx 0.56$, $k \approx 22.4$;
if $n = 13$, then $\alpha \approx 1.44$, $k \approx 10.0$;
if $n = 14$, then $\alpha \approx 2.37$, $k \approx 6.9$.

3 On power-law asymptotic behavior of solutions to weakly superlinear Emden–Fowler type equations with constant potential

It appears that a weaker version of the I. T. Kiguradze's hypothesis concerning power-law asymptotic behavior of blow-up solutions for higher-order equations (2) can be proved. A sketch of the proof is contained in [7].

Theorem 2. *For any integer* $n > 4$ *there exists* $K > 1$ *such that for any real* $k \in (1, K)$, *all* n*-positive solutions of equation* (2) *have the power-law asymptotic behavior* (5) *near the right endpoints of their domains.*

References

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