Qualitative properties of lattice reaction-diffusion equations

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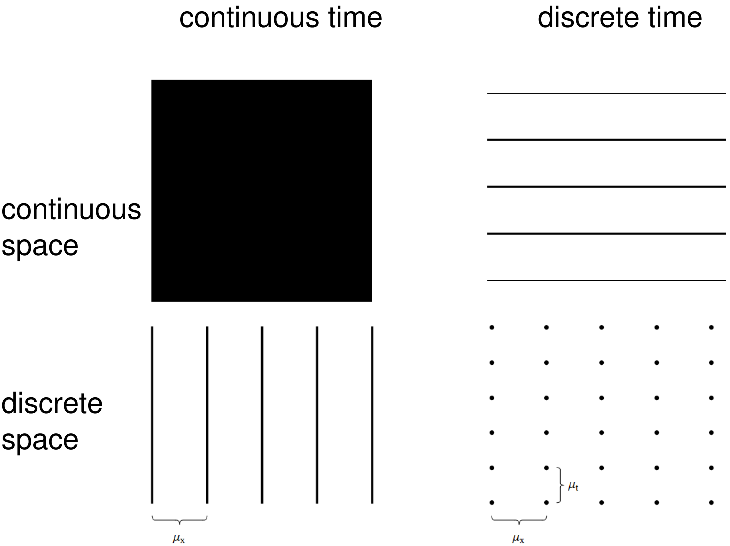
Joint work with

- Antonín Slavík,
- Jonáš Volek,
- Michal Friesl.

Underlying structures in evolutionary PDEs



Mathematical Motivation for Discrete-Space PDEs

- Spatial discretization
- Random walks
- Transition problems

Motivation: PDE (semi)discretization

- Spatial discretization of the classical diffusion/heat equation:

$$\frac{\partial u}{\partial t}(x,t) = a\frac{\partial^2 u}{\partial x^2}(x,t)$$

$$\Downarrow$$

$$u_t(x,t) = au(x+1,t) - 2au(x,t) + au(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{R}$$

E. Rothe, *Zweidimensionale parabolische randwertaufgaben als grenzfall eindimensionaler randwertaufgaben*, Mathematische Annalen 102 (1930), 650–670.

Motivation: Random walks on $\mathbb{Z}$

One-dimensional random walk on $\mathbb{Z}$ with discrete time

Transition probabilites: $a, b, c \in [0, 1]$, $a + b + c = 1$

$$u(x, t) = \text{probability of visiting } x \text{ at time } t$$

$$u(x, t+1) = au(x+1, t) + bu(x, t) + cu(x-1, t), \quad x \in \mathbb{Z},\ t \in \mathbb{N}_0$$

$$\Delta_t u(x, t) = au(x+1, t) + (b-1)u(x, t) + cu(x-1, t), \quad x \in \mathbb{Z},\ t \in \mathbb{N}_0$$

Motivation: Transition problems

$$\frac{\partial u}{\partial t} = a\frac{\partial^2 u}{\partial x^2} + f(u)$$

$$\Downarrow$$

$$u_t(x,t) = au(x+1,t) - 2au(x,t) + au(x-1,t) + f(u),$$

- Transition between discrete and continuous problems,
- Transition between ODEs and PDEs,
  - Infinite system of ODEs,
  - ODEs in sequence spaces.
- Local x spatial dynamics,
  - diffusion - spatial dynamics,
  - reaction function - local dynamics.

Application Motivation for Discrete-Space PDEs

- Image processing

  T. Lindeberg, *Scale-space for discrete signals*, IEEE Transactions on Pattern Analysis and Machine Intelligence 12 (1990), no. 3, 234–254.

- Material sciences

  J. W. Cahn, *Theory of Crystal Growth and Interface Motion in Crystalline Materials*, Acta.Metall. 8 (1960), 87–118.

- Biology

  J. Campbell, *The SMM model as a boundary value problem using the discrete diffusion equation*, Theoretical Population Biology 72 (2007), no. 4, 539–546.

- (networks, electrical circuits...)

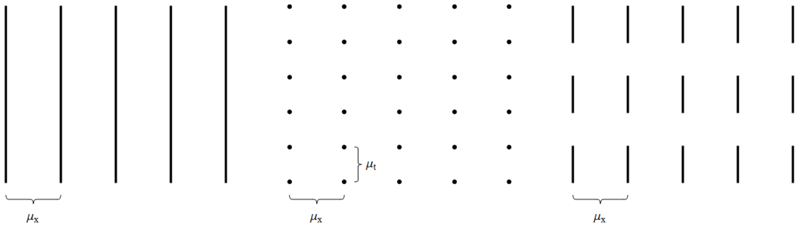
Linear case

Diffusion equation on lattices

Underlying structures

Our motivation

- (non)linear diffusion on lattices,
- different time structures, convergence.



Linear diffusion equations on lattices

We consider a class of partial dynamic equations with discrete space and arbitrary (continuous, discrete or mixed) time:

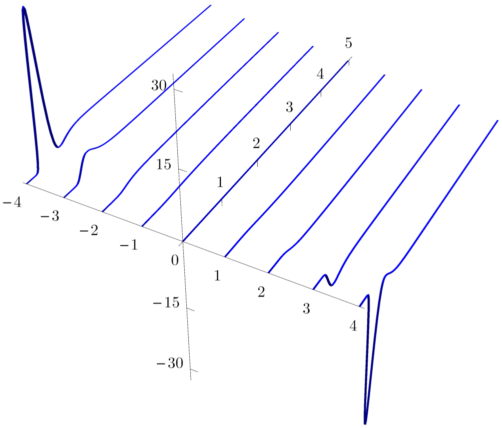
$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{T}$$

- $\mathbb{T}$ is a time scale (arbitrary closed subset of $\mathbb{R}$)
- a, b, $c \in \mathbb{R}$
- $u^{\Delta}(x,t)$ is the Δ-derivative of u with respect to t

A. Slavík, P. Stehlík
Explicit solutions to dynamic diffusion-type equations and their time integrals.
Applied Mathematics and Computations 234(2014), 486–505.

Existence and uniqueness for IVPs (1)

In general, initial-value problems do not have a unique forward solution ($\mathbb{T} = \mathbb{R}$); we get uniqueness by restricting ourselves to the class of bounded solutions.



Existence and uniqueness for IVPs (2)

Bounded backward solutions need not exist or be unique; additional assumption on the time scale graininess is necessary.

Theorem

Consider an interval $[T_1, T_2]_{\mathbb{T}} \subset \mathbb{T}$ and a point $t_0 \in [T_1, T_2]_{\mathbb{T}}$. Let $u^0 \in \ell^\infty(\mathbb{Z})$. Assume that $\mu(t) < \frac{1}{|a|+|b|+|c|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$. Then

$$u^\Delta(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t), \quad x \in \mathbb{Z},\ t \in \mathbb{Z}$$

has a unique bounded solution on $\mathbb{Z} \times [T_1, T_2]_{\mathbb{T}}$ satisfying $u(x, t_0) = u^0_x$ for every $x \in \mathbb{Z}$.

Explicit solutions – examples

Using generating functions we can derive, e.g.:

- $\mathbb{T} = \mathbb{R}$:

$$u(x,t) = e^{bt} I_x(2t\sqrt{ac}) \left(\sqrt{\frac{c}{a}}\right)^x$$

- $\mathbb{T} = \mathbb{Z}$:

$$u(x,t) = \sum_{j=0}^{t} \binom{t}{j,\, t-2j-x,\, j+x} a^j (b+1)^{t-2j-x} c^{j+x}$$

- $\mathbb{T} = \{H_n,\ n \in \mathbb{N}_0\}$, where $H_0 = 0$ and $H_n = \sum_{k=1}^{n} \frac{1}{k}$:

$$u(x,H_n) = \frac{1}{n!} \sum_{l=|x|}^{n} \sum_{j=0}^{l} s(n,l) \binom{l}{j,\, l-2j-x,\, j+x} a^j (b+n)^{l-2j-x} c^{j+x}$$

Sum-preserving RHS

We consider the problem

$$u^{\Delta_t}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t).$$

Theorem

Let $u : \mathbb{Z} \times [T_1, T_2]_{\mathbb{T}} \to \mathbb{R}$ be a loc.bounded solution and $a + b + c = 0$. Assume that:

- *For a certain $t_0 \in [T_1, T_2]_{\mathbb{T}}$, the sum $\sum_{x\in\mathbb{Z}} |u(x,t_0)|$ is finite.*
- *$\mu(t) < \frac{1}{|a|+|b|+|c|}$ for every $t \in [T_1, t_0)_{\mathbb{T}}$.*

Then $\sum_{x\in\mathbb{Z}} u(x,t) = \sum_{x\in\mathbb{Z}} u(x,t_0)$ for every $t \in [T_1, T_2]_{\mathbb{T}}$.

Counterexample

The condition $\mu(t) < \frac{1}{|a|+|b|+|c|}$ cannot be omitted. Consider, $a = c = 1$, $b = -2$

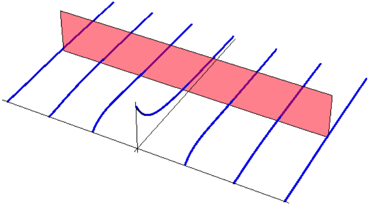
$$\begin{cases} u^{\Delta}(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t), \quad x \in \mathbb{Z}, t \in \frac{1}{4}\mathbb{Z}, \\ u(x,0) = 0. \end{cases}$$

$u(x,-1/4) = (-1)^x$

Stochastic processes

If $\mu(t) < -1/b$ then for forward solutions

- sign is preserved,
- space sums are preserved,

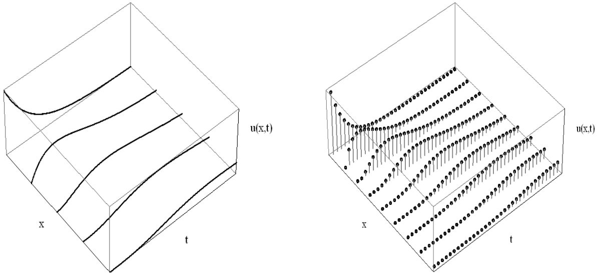


Thus, we talk about dynamic stochastic processes.

Stehlik P., Volek J.
Transport equation on semidiscrete domains and Poisson-Bernoulli processes.
Journal of Difference Equations and Applications. 2013, 19:3, 439–456.

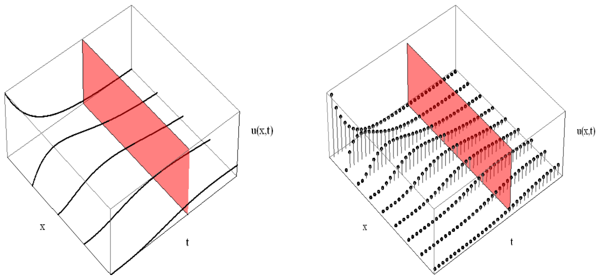
M. Friesl, A. Slavík, P. Stehlík
Discrete-space partial dynamic equations on time scales and applications to stochastic processes.
Applied Mathematics Letters 37 (2014), 86–90.

Counting stochastic processes



- $f_t(x) = u(x, t)$ - probability of number of events (occurrences) until time t,
- $g_0(t) = u(0, t)$ - probability distribution of the time of the first occurrence,
- $g_x(t) = u(x - 1, \cdot)$ probability distributions that x events have happened until time t,
- moreover, $u(0, t)$ - waiting time until the next occurrence.

Counting stochastic processes



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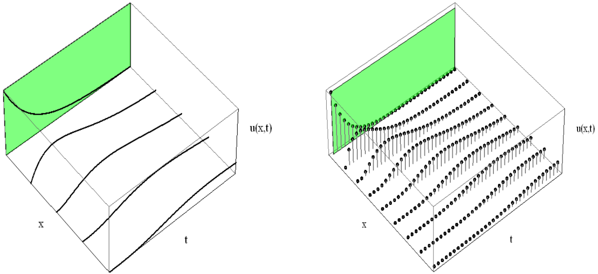

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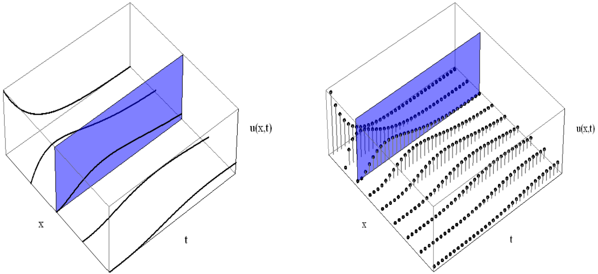

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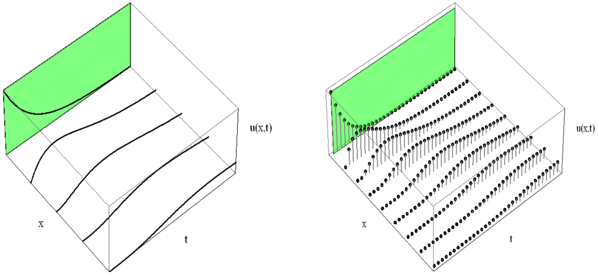

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- moreover, $u(0, t)$ - waiting time until the next occurrence.

Counting stochastic processes

	$f_t(x)$	$g_0(t)$	$g_x(t), x \geq 0$
$\mathbb{Z} \times \mathbb{R}$	Poisson dist.	exponential dist.	Erlang (Gamma) dist.
$\mathbb{Z} \times p\mathbb{Z}$	binomial dist.	geometric dist.	negative binomial dist.

Simeon Denis Poisson
(1781-1840)

Jacob Bernoulli
(1654-1705)

Example - Heterogeneous Bernoulli Process

p_i - probability of success in i-th trial(in contrast to standard Bernoulli process non-constant)

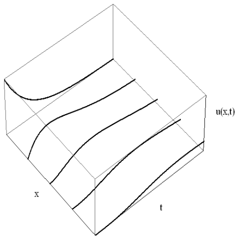
$$\mathbb{T} = \left\{ 0, p_1, p_1 + p_2, \ldots, \sum_{i=1}^{n-1} p_i, \ldots \right\},$$

For illustration, let us consider 3 cases

1. Bernoulli case $p_i = \frac{1}{2}$, (dice rolling)
2. decreasing probability case $p_i = \frac{1}{i}$, (jumping over an obstacle)
3. increasing probability case $p_i = \frac{i-1}{i}$. (exam success)

Time integrals/sums

In general, time integrals are not preserved.



We observe time integrals preservation only in very special cases - transport equation (e.g. $a = 0$ and $b = c$)
We focus on the more general question:

Under which condition are the time integrals/sums finite?

Time integrals/sums

- Difficult to analyze.
- We use explicit solutions:

$$u(x,t) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} \binom{k}{l, k-2l-x, l+x} a^l b^{k-2l-x} c^{l+x} \right) h_k(t, t_0), \quad x \in \mathbb{Z}$$

- $\mathbb{T} = \mathbb{R}$:

$$u(x,t) = e^{bt} I_x(2t\sqrt{ac}) \left(\sqrt{\frac{c}{a}} \right)^x$$

- $\mathbb{T} = \mathbb{Z}$:

$$u(x,t) = \sum_{j=0}^{t} \binom{t}{j, t-2j-x, j+x} a^j (b+1)^{t-2j-x} c^{j+x}$$

Exact integrals

Theorem

Let u be the unique locally bounded solution with $a, c > 0$, $a \neq c$, and $a + b + c = 0$.

- *If $c > a$, then*

$$\int_0^\infty u(x,t)\,\Delta t = \begin{cases} \frac{\left(\frac{c}{a}\right)^x}{c-a} & \text{if } x < 0, \\ \frac{1}{c-a} & \text{if } x \geq 0. \end{cases}$$

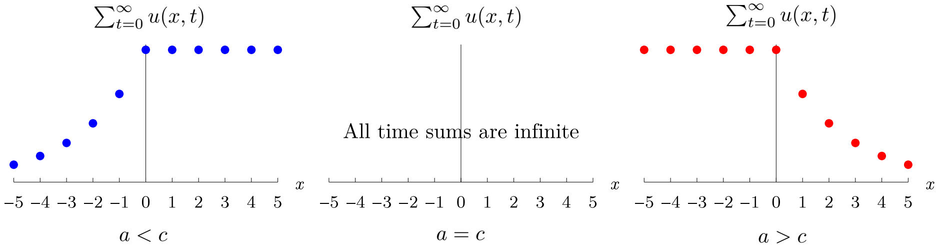
- *If $c < a$, then*

$$\int_0^\infty u(x,t)\,\Delta t = \begin{cases} \frac{1}{a-c} & \text{if } x \leq 0, \\ \frac{\left(\frac{c}{a}\right)^x}{a-c} & \text{if } x > 0. \end{cases}$$

Exact integrals/sums - illustration

Surprisingly

- time integrals are constant in one direction,
- the values are independent of the underlying time scales.



Going nonlinear

Reaction-diffusion equation

Reaction-diffusion equation

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t), \quad x \in \mathbb{Z}, \quad t \in \mathbb{T}$$

- naturally, no explicit solutions,
- qualitative questions
  - existence,
  - uniqueness,
  - continuous dependence,
  - maximum principles.

Assumptions on the reaction function

$$u^{\Delta}(x,t) = au(x+1,t) + bu(x,t) + cu(x-1,t) + f(u(x,t),x,t),$$

Assumptions on $f : \mathbb{R} \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$:

(H1) *f is bounded on each set $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.*

(H2) *f is Lipschitz-continuous in the first variable on each set $B \times \mathbb{Z} \times [t_0, T]_{\mathbb{T}}$, where $B \subset \mathbb{R}$ is bounded.*

(H3) *For each bounded set $B \subset \mathbb{R}$ and each choice of $\varepsilon > 0$ and $t \in [t_0, T]_{\mathbb{T}}$, there exists a $\delta > 0$ such that if $s \in (t-\delta, t+\delta) \cap [t_0, T]_{\mathbb{T}}$, then $|f(u,x,t) - f(u,x,s)| < \varepsilon$ for all $u \in B$, $x \in \mathbb{Z}$.*

Abstract formulation

Studying the abstract problem in ℓ^∞:

$$U^\Delta(t) = \Phi(U(t), t),$$

with $U : [t_0, t_0 + \delta]_\mathbb{T} \to \ell^\infty(\mathbb{Z})$ and $\Phi : \ell^\infty(\mathbb{Z}) \times [t_0, T]_\mathbb{T} \to \ell^\infty(\mathbb{Z})$ being given by

$$\Phi(\{u_x\}_{x\in\mathbb{Z}}, t) = \{au_{x+1} + bu_x + cu_{x-1} + f(u_x, x, t)\}_{x\in\mathbb{Z}},$$

we get

- Uniqueness,
- Local existence (bounded time interval),
- Continuous dependence on initial condition,
- Continuous dependence on the underlying time scale.

Weak Maximum principle

Additional assumptions on f:

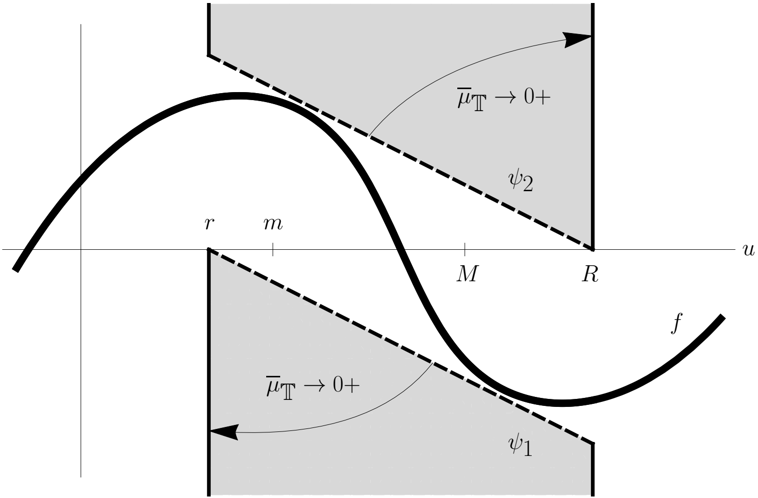
(*H4*) *$a, b, c \in \mathbb{R}$ are such that $a, c \geq 0$, $b < 0$, and $a + b + c = 0$.*

(*H5*) *$b < 0$ and $\overline{\mu}_{\mathbb{T}} \leq -1/b$.*

(*H6*) *There exist $r, R \in \mathbb{R}$ such that $r \leq m \leq M \leq R$, and one of the following statements holds:*

- $\overline{\mu}_{\mathbb{T}} = 0$ *and* $f(R, x, t) \leq 0 \leq f(r, x, t)$ *for all* $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.
- $\overline{\mu}_{\mathbb{T}} > 0$ *and* $\dfrac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(r - u) \leq f(u, x, t) \leq \dfrac{1 + \overline{\mu}_{\mathbb{T}} b}{\overline{\mu}_{\mathbb{T}}}(R - u)$ *for all* $u \in [r, R]$, $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.

Illustration - key assumption on f



Weak Maximum Principle

Theorem (weak maximum principle)

Assume that (H1)–(H6) hold. If $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ is a bounded solution of RDE, then

$$r \leq u(x,t) \leq R \quad \textit{for all} \quad x \in \mathbb{Z}, \quad t \in [t_0, T]_{\mathbb{T}}.$$

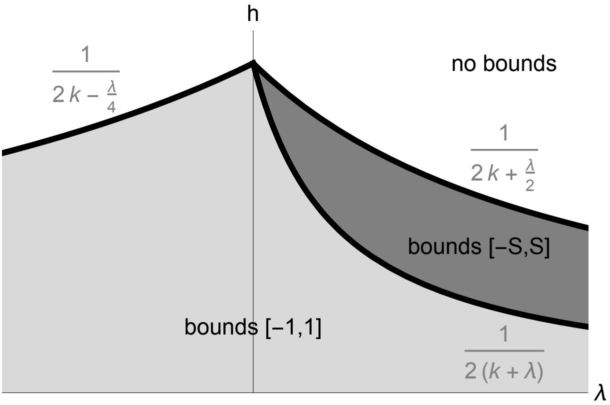
Corollary - global existence and continuous dependence

Theorem (global existence)

If $u^0 \in \ell^\infty(\mathbb{Z})$ and (H1)–(H6) hold, then RDE has a unique bounded solution $u : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$.
Moreover, the solution depends continuously on u^0 in the following sense: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that if $v^0 \in \ell^\infty(\mathbb{Z})$, $r \leq v^0_x \leq R$ for all $x \in \mathbb{Z}$, and $\|u^0 - v^0\|_\infty < \delta$, then the unique bounded solution $v : \mathbb{Z} \times [t_0, T]_{\mathbb{T}} \to \mathbb{R}$ of RDE corresponding to the initial condition v^0 satisfies $|u(x,t) - v(x,t)| < \varepsilon$ for all $x \in \mathbb{Z}$, $t \in [t_0, T]_{\mathbb{T}}$.

Lattice Nagumo equation

$$u^{\Delta} = k\Delta^2 u(x-1,t) + \lambda u\left(1-u^2\right),$$
$$u(x,t_0) = u_x^0, \quad x \in \mathbb{Z}.$$



Děkuji za pozornost

გმადლობთ

ყურადღებისთვის