Non-negative periodic solutions of second-order differential equations with sublinear nonlinearities

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- $p\colon [0,\omega] \to \mathbb{R}$ ... Lebesgue integrable
- $q\colon [0,\omega] \times \mathbb{R} \to \mathbb{R}$ ... Carathéodory + sublinear

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$$\left.\begin{array}{l} |q(t,x)| \le q_0(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \ge x_0, \\ x_0 > 0, \quad q_0\colon [0,\omega] \times [x_0,+\infty[\to [0,+\infty[\text{ is a Carathéodory function,} \\ q_0(t,\cdot)\colon [x_0,+\infty[\to [0,+\infty[\text{ is non-decreasing for a. e. } t \in [0,\omega], \\ \displaystyle\lim_{x\to+\infty} \frac{1}{x}\int_0^\omega q_0(s,x)\mathrm{d}s = 0. \end{array}\right\} \tag{H_1}$$

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$$\left.\begin{array}{l} q(t,x) \geq x g(t,x) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in]0,\delta], \\ \delta > 0, \quad g\colon [0,\omega] \times]0,\delta] \to \mathbb{R} \text{ is a locally Carathéodory function,} \\ g(t,\cdot)\colon]0,\delta] \to \mathbb{R} \text{ is non-increasing for a. e. } t \in [0,\omega], \end{array}\right\} \qquad (H_2)$$

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$$\left.\begin{array}{l} \text{for every } b > a > 0, \text{ there exists } h_{ab} \in L([0,\omega]) \text{ such that} \\ h_{ab}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \quad h_{ab} \not\equiv 0, \\ q(t,x) \geq h_{ab}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [a,b], \end{array}\right\} \tag{H_3}$$

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$$\left.\begin{array}{l} \text{For every } b > a > 0 \text{ and } c > 0, \text{ there exists } h_{abc} \in L([0,\omega]) \text{ such that} \\ h_{abc}(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega],\ h_{abc} \not\equiv 0, \\ \dfrac{q(t,x)}{x} - \dfrac{q(t,x+c)}{x+c} \geq h_{abc}(t) \quad \text{for a. e. } t \in [0,\omega] \text{ and all } x \in [a,b]. \end{array}\right\} \tag{H_4}$$

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▷ particular case:

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

- $p, h \in L([0,\omega])$
- $\lambda \in \,]0,1[$

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- $p, h \in L([0,\omega])$
- $\lambda \in]0,1[$

either $h \geq 0$ a. e. on $[0,\omega]$, or $h \leq 0$ a. e. on $[0,\omega]$,

the coefficient p is not constant and can change its sign!!!

$$\boxed{u'' = p(t)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)}$$

- We say that $\boxed{p \in \mathcal{V}^+(\omega)}$ if

$$\left.\begin{array}{l} u \in AC^1([0,\omega]), \\ u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \\ u(0) = u(\omega),\ u'(0) = u'(\omega) \end{array}\right\} \implies \quad u(t) \geq 0 \quad \text{for } t \in [0,\omega].$$

Alternatively – Green's function is positive, or antimaximum principle holds

- We say that $\boxed{p \in \mathcal{V}^-(\omega)}$ if

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Alternatively – Green's function is negative, or maximum principle holds

$$\boxed{u'' = p(t)u + f(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)}$$

- We say that $\boxed{p \in \mathcal{V}_0(\omega)}$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)$$

has a positive solution.

- We say that $\boxed{p \in \mathcal{D}_1(\omega)}$ if for any $\alpha \in [0, \omega[$, the solution u of the problem the problem

$$u'' = \widetilde{p}(t)u; \quad u(\alpha) = 0,\ u'(\alpha) = 1$$

has at most one zero on the interval $]\alpha, \alpha + \omega[$, where $\widetilde{p}$ is the ω-periodic extension of p to the whole real axis.

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \; u'(0) = u'(\omega) \tag{1}$$

- $h(t) \geq 0$ for a. e. $t \in [0, \omega]$, $h \not\equiv 0$

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$$y'' = ay + b\sqrt[3]{y}$$

- $b > 0$

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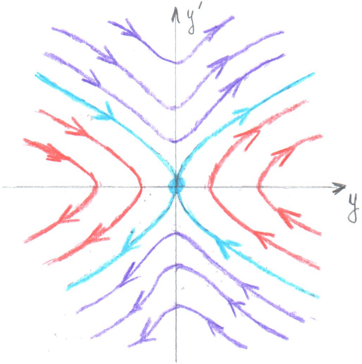

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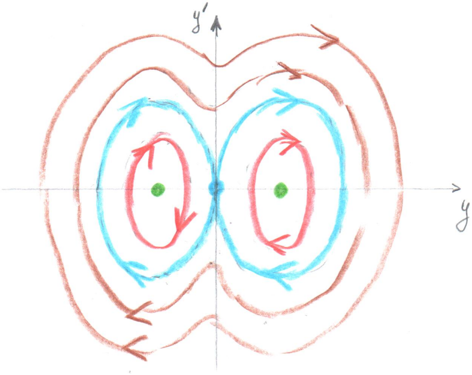

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Theorem. Let $\lambda \in]0, 1[$ and

$$h(t) > 0 \quad \text{for a. e. } t \in [0, \omega]. \tag{A_1}$$

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Example. Consider a particular case of (1) with

$$p(t) := -1, \quad h(t) := 3(1 - \sin t) \quad \text{for } t \in [0, 2\pi], \qquad \lambda := \frac{1}{2}\,, \qquad \omega := 2\pi,$$

namely, the problem

$$\boxed{u'' = -u + 3(1 - \sin t)\sqrt{|u|} \operatorname{sgn} u; \quad u(0) = u(2\pi),\ u'(0) = u'(2\pi).} \tag{2}$$

Then $p \in \mathcal{D}_1(\omega)$, $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)$, (A_1) holds, and problem (2) has a solution

$$u(t) := (1 + \sin t)^2 \quad \text{for } t \in [0, 2\pi].$$

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(2a) $p \in \mathcal{V}^+(\omega) \quad \Rightarrow \quad$ (1) has exactly 3 solutions (> 0, < 0, $\equiv 0$)

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Remark: Assertions (1) and (2a) remain true even if (A_1) is relaxed to

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0,\omega], \qquad h \not\equiv 0. \tag{A_2}$$

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Open questions:

- $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega) \quad \Rightarrow \quad$ (1) has a positive solution?
- $p \notin \mathcal{D}_1(\omega) \quad \Rightarrow \quad$ (1) has a sign-changing solution?

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

▷ $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \Rightarrow \quad$ (1) has at least one non-trivial non-negative solution

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  $\Rightarrow \quad \exists$ an arbitrarily large positive lower function α of problem (1)
- $\exists r > 0$ such that $p + \frac{h}{r^{1-\lambda}} \in \mathcal{V}^-(\omega)$

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- $\delta > 0$ large enough, cutting function $\chi(x) := [x]_+ - [x - \delta]_+$ for $x \in \mathbb{R}$

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- $\delta > 0$ large enough, cutting function $\chi(x) := [x]_+ - [x - \delta]_+$ for $x \in \mathbb{R}$
- auxiliary problem

$$u'' = (p(t) + \ell(t))u + h(t)|\chi(u)|^{\lambda} \operatorname{sgn} \chi(u) - \ell(t)\chi(u); \quad \text{PBC} \tag{3}$$

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

▷ $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \Rightarrow \quad$ (1) has at least one non-trivial non-negative solution

- $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \Rightarrow \quad \exists \ell \in L([0,\omega])$, such that $\ell \geq 0$ and $p + \ell \in \operatorname{Int} \mathcal{V}^+(\omega)$ $\Rightarrow \quad \exists$ an arbitrarily large positive lower function α of problem (1)
- $\exists r > 0$ such that $p + \frac{h}{r^{1-\lambda}} \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad \exists$ an arbitrarily small positive upper function β of problem (1)
- $\delta > 0$ large enough, cutting function $\chi(x) := [x]_+ - [x - \delta]_+$ for $x \in \mathbb{R}$
- auxiliary problem

$$u'' = (p(t) + \ell(t))u + h(t)|\chi(u)|^\lambda \operatorname{sgn} \chi(u) - \ell(t)\chi(u); \quad \text{PBC} \tag{3}$$

- (α, β) is a couple of reverse-ordered lower and upper functions of (3)

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- (5) $\quad \Rightarrow \quad u \not\equiv 0$
- $u(t) \leq \delta$ for $t \in [0, \omega] \quad \Rightarrow \quad \chi(u) \equiv u \quad \Rightarrow \quad u$ is a non-trivial non-negative solution of (1)

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda}\,\mathrm{sgn}\,u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

▷ $p \notin \mathcal{V}^{-}(\omega) \cup \mathcal{V}_0(\omega) \quad \Rightarrow \quad$ (1) has at least one non-trivial non-negative solution

▷ $p \in \mathcal{D}_1(\omega)$ and u is a solution of (1) $\quad \Rightarrow$

$$\text{either} \quad u(t) \geq 0 \quad \text{for } t \in [0, \omega], \quad \text{or} \quad u(t) \leq 0 \quad \text{for } t \in [0, \omega]$$

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \qquad (1)$$

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▷ $p \in \mathcal{V}^+(\omega)$ and u is a solution of (1) $\quad \Rightarrow$

$$\text{either} \quad u(t) > 0 \quad \text{for } t \in [0,\omega], \quad \text{or} \quad u(t) \leq 0 \quad \text{for } t \in [0,\omega]$$

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

- $h(t) \leq 0$ for a. e. $t \in [0, \omega]$, $h \not\equiv 0$

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

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$$\boxed{y'' = ay + b\sqrt[3]{y}}$$

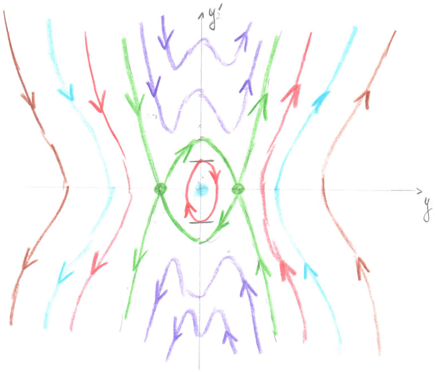
- $b < 0$

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

- $h(t) \leq 0$ for a. e. $t \in [0, \omega]$, $h \not\equiv 0$

$$\boxed{y'' = ay + b\sqrt[3]{y}}$$

- $b < 0$, $a > 0$



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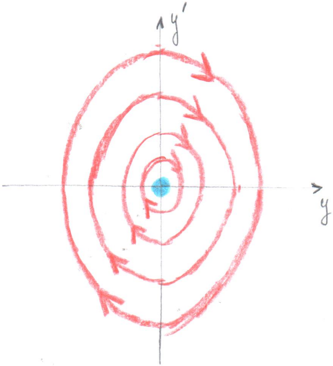

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda}\,\mathrm{sgn}\,u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

Theorem. Let $\lambda \in]0,1[$ and

$$h(t) \leq 0 \quad \text{for a. e. } t \in [0,\omega], \qquad h \not\equiv 0. \tag{A_3}$$

Then:

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Theorem. Let $\lambda \in]0,1[$ and

$$h(t) \leq 0 \quad \text{for a. e. } t \in [0,\omega], \qquad h \not\equiv 0. \tag{A_3}$$

Then:

(1) $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has exactly 3 solutions (> 0, < 0, $\equiv 0$)

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Open questions:

- $p \notin \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has a non-trivial non-negative solution?
- $p \notin \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has a sign-changing solution?

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda}\,\mathrm{sgn}\,u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

▷ $p \in \mathcal{V}^{-}(\omega) \quad \Rightarrow \quad$ (1) has at least one positive solution

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda}\,\mathrm{sgn}\,u; \quad u(0) = u(\omega),\; u'(0) = u'(\omega)} \qquad (1)$$

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- $p \in \mathcal{V}^{-}(\omega) \quad \Rightarrow \quad \exists r > 0$ such that $p - \frac{h}{r^{1-\lambda}} \in \mathrm{Int}\,\mathcal{V}^{+}(\omega)$

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- (α, β) is a couple of well-ordered lower and upper functions of (1)

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- (α, β) is a couple of well-ordered lower and upper functions of (1)
- (1) has a solution u such that

$$0 < \alpha(t) \leq u(t) \leq \beta(t) \qquad \text{for } t \in [0, \omega] \tag{5}$$

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▷ $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has at least one positive solution

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- assume the contrary: (1) has two distinct positive solutions

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega), \ u'(0) = u'(\omega) \tag{1}$$

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- $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has solutions u, v such that

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$$u(t) < v(t) \quad \text{for } t \in [0, \omega].$$

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(1) Assume that

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- u, v are positive periodic solutions, respectively, to equations

$$z'' = \big(p(t) + h(t)v^{\lambda-1}(t)\big)\,z + h(t)\big[u^{\lambda-1}(t) - v^{\lambda-1}(t)\big]\,u(t)$$
$$z'' = \big(p(t) + h(t)v^{\lambda-1}(t)\big)\,z$$

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

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$$z'' = \big(p(t) + h(t)v^{\lambda-1}(t)\big)z$$

- the third Fredholm's theorem $\quad \Rightarrow$

$$0 = \int_0^\omega h(s)\big[u^{\lambda-1}(s) - v^{\lambda-1}(s)\big]u(s)v(s)\mathrm{d}s \leq Const. \int_0^\omega h(s)\mathrm{d}s < 0$$

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(2) Assume that

$$u(t_*) = v(t_*) \quad \text{for some } t_* \in [0, \omega].$$

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

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(2) Assume that

$$u(t_*) = v(t_*) \quad \text{for some } t_* \in [0, \omega].$$

- $w(t) := u(t) - v(t)$ is a solution of the problem

$$w'' = p(t)w + h(t)\big[u^{\lambda}(t) - v^{\lambda}(t)\big]$$

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

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$$w'' = p(t)w + h(t)\big[u^\lambda(t) - v^\lambda(t)\big]$$

- if $h(\cdot)\big[u^\lambda(\cdot) - v^\lambda(\cdot)\big] \equiv 0$, then $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad w \equiv 0$ – contradiction

$$\boxed{u'' = p(t)u + h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

▷ $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has at least one positive solution

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- $w(t) := u(t) - v(t)$ is a solution of the problem

$$w'' = p(t)w + h(t)\big[u^\lambda(t) - v^\lambda(t)\big]$$

- if $h(\cdot)\big[u^\lambda(\cdot) - v^\lambda(\cdot)\big] \equiv 0$, then $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad w \equiv 0$ – contradiction
- if $h(\cdot)\big[u^\lambda(\cdot) - v^\lambda(\cdot)\big] \not\equiv 0$, then $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad w(t) < 0$ on $t[0, \omega]$ – contradiction

$$\boxed{u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u; \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \tag{1}$$

▷ $p \in \mathcal{V}^-(\omega) \quad \Rightarrow \quad$ (1) has at least one positive solution

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