Asymptotic formulae for solutions of half-linear differential equations

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Structure of the talk

- Regular variation
  - Karamata theory
  - De Haan theory
- Asymptotic properties of half-linear differential equations
  - Solutions in the class Γ
  - Solutions in the class Π
  - More on regularly varying solutions
- Remarks, comments

Theory of regularly varying functions

- initiated by J. Karamata (1930). But there are also earlier works ...
- study of relations such that

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = g(\lambda) \in (0, \infty), \quad \forall \lambda > 0,$$

together with their applications (integral transforms – Tauberian theorems, probability theory, number theory, complex analysis, differential equations, etc.)

N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, 1987.

J. L. Geluk, L. de Haan, *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract 40, Amsterdam, 1987.

V. Marić, Regular Variation and Differential Equations, Lecture Notes in Mathematics 1726, Springer-Verlag, Berlin-Heidelberg-New York, 2000.

E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics 508, Springer-Verlag, Berlin-Heidelberg-New York, 1976.

Definition

A measurable function $f : [a, \infty) \to (0, \infty)$ is called regularly varying (at ∞) of index ϑ if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \text{ for all } \lambda > 0.$$

[Notation: $f \in \mathcal{RV}(\vartheta)$]
If $\vartheta = 0$, then f is called slowly varying.

[Notation: $f \in \mathcal{SV}$]

[$\mathcal{RV}_0$ means regular variation at zero.]

The Uniform Convergence Theorem

If f $\in \mathcal{RV}(\vartheta)$, then the relation in the definition holds uniformly on each compact λ-set in $(0, \infty)$.

Representation Theorem

- *f is regularly varying of index ϑ if and only if*

$$f(t) = \varphi(t) \exp\left\{ \int_a^t \frac{\delta(s)}{s}\, ds \right\}$$

where $\varphi(t) \to const > 0$ and $\delta(t) \to \vartheta$ as $t \to \infty$.

- *f is regularly varying of index ϑ if and only if*

$$f(t) = t^{\vartheta} \varphi(t) \exp\left\{ \int_a^t \frac{\psi(s)}{s}\, ds \right\}$$

where $\varphi(t) \to const > 0$ and $\psi(t) \to 0$ as $t \to \infty$.

If $\varphi(t) \equiv$ const, then f is said to be normalized regularly varying ($f \in \mathcal{NRV}$).

Examples of (non-)$\mathcal{SV}$ functions

f is regularly varying of index ϑ if and only if

$$f(t) = t^{\vartheta} L(t),$$

where $L \in \mathcal{SV}$.

- $\prod_{i=1}^{n} (\ln_i t)^{\mu_i}$, where $\ln_i t = \ln \ln_{i-1} t$ and $\mu_i \in \mathbb{R}$ is $\mathcal{SV}$ function.
- $2 + \sin(\ln_2 t)$ and $(\ln \Gamma(t))/t$ are $\mathcal{SV}$ functions.
- $\frac{1}{t} \int_a^t \frac{1}{\ln s} \, ds$ is $\mathcal{SV}$ function.
- SV functions may exhibit "infinite oscillation" (i.e., $\liminf_{t \to \infty} L(t) = 0$, $\limsup_{t \to \infty} L(t) = \infty$), for example, $\exp\left\{(\ln t)^{\frac{1}{3}} \cos(\ln t)^{\frac{1}{3}}\right\}$.
- $2 + \sin t$, $2 + \sin(\ln t)$ are NOT $\mathcal{SV}$ functions.
- $\exp t$ is NOT $\mathcal{RV}$ function.

- Extension in a logical and useful manner of the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies "more slowly" than a power function.
- $\mathcal{SV} \subset \mathcal{RV}$, but $\mathcal{SV}$ vs. $\mathcal{RV}(\vartheta)$ with $\vartheta \neq 0$
- Regularly varying functions have a "good behavior" with respect to integration resp. summation.
- Regularly varying functions and related objects naturally occur in differential equations.
- ...

Other selected properties

- If $L_1, \dots, L_n \in \mathcal{SV}$, $n \in \mathbb{N}$, and $R(x_1, \dots, x_n)$ is a rational function with non-negative coefficients, then $R(L_1, \dots, L_n) \in \mathcal{SV}$. In particular,

$$f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2) \text{ and } f_1^\gamma \in \mathcal{RV}(\gamma\vartheta_1)$$

  for $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, and $\gamma \in \mathbb{R}$. Moreover, $L_1 \circ L_2 \in \mathcal{SV}$ provided $L_2(t) \to \infty$ as $t \to \infty$.

- If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^\vartheta L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.

- If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta \le 0$ and $f(t) = \int_t^\infty g(s)\, ds$ with g nonincreasing, then

$$\frac{-tf'(t)}{f(t)} = \frac{tg(t)}{f(t)} \to -\vartheta \text{ as } t \to \infty.$$

- If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta \ge 0$ and $f(t) = f(t_0) + \int_{t_0}^t g(s)\, ds$ with g monotone, then

$$\frac{tf'(t)}{f(t)} = \frac{tg(t)}{f(t)} \to \vartheta \quad \text{as } t \to \infty.$$

Other selected properties

- *(Almost monotonicity)* For a positive measurable function L it holds: $L \in \mathcal{SV}$ if and only if, for every $\vartheta > 0$, there exist a (regularly varying) nondecreasing function F and a (regularly varying) nonincreasing function G with
$$t^{\vartheta} L(t) \sim F(t) \quad t^{-\vartheta} L(t) \sim G(t) \qquad \text{as } t \to \infty.$$
- *(Asymptotic inversion)* If $g \in \mathcal{RV}(\vartheta)$ with $\vartheta > 0$, then there exists $g \in \mathcal{RV}(1/\vartheta)$ with
$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as } t \to \infty.$$
Here g (an "asymptotic inverse" of f) is determined uniquely to within asymptotic equivalence. One version of g is the generalized inverse $f^{\leftarrow}(t) := \inf\{s \in [a, \infty) : f(s) > t\}$.

Other selected properties (Karamata's theorem!!)

- *(Karamata's theorem; direct half)* If $L \in \mathcal{SV}$, then

$$\int_t^\infty s^\gamma L(s)\,ds \sim \frac{1}{-\gamma-1} t^{\gamma+1} L(t)$$

provided $\gamma < -1$, and

$$\int_a^t s^\gamma L(s)\,ds \sim \frac{1}{\gamma+1} t^{\gamma+1} L(t)$$

provided $\gamma > -1$. The integral $\int_a^\infty L(s)/s\,ds$ may or may not converge. The function $\tilde{L}(t) = \int_a^t L(s)/s\,ds$ is a new SV function and such that $L(t)/\tilde{L}(t) \to 0$ as $t \to \infty$.

- *(Karamata's theorem; converse half)* If for some $\sigma < -(\zeta+1)$,

$$t^{\sigma+1} f(t) \Big/ \int_t^\infty s^\sigma f(s)\,ds \to -(\sigma+\zeta+1) \text{ as } t \to \infty, \text{ then } f \in \mathcal{RV}(\zeta).$$

If for some $\sigma > -(\zeta+1)$,

$$t^{\sigma+1} f(t) \Big/ \int_a^t s^\sigma f(s)\,ds \to \sigma+\zeta+1 \text{ as } t \to \infty, \text{ then } f \in \mathcal{RV}(\zeta).$$

Rapid variation

Definition

A measurable function $f : [a, \infty) \to (0, \infty)$ is called rapidly varying of index ∞, we write $f \in \mathcal{RPV}(\infty)$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} 0 & \text{for } 0 < \lambda < 1, \\ \infty & \text{for } \lambda > 1, \end{cases}$$

and is called rapidly varying of index $-\infty$, we write $f \in \mathcal{RPV}(-\infty)$, if

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = \begin{cases} \infty & \text{for } 0 < \lambda < 1, \\ 0 & \text{for } \lambda > 1. \end{cases}$$

The class of all rapidly varying solutions is denoted as $\mathcal{RPV}$.

While $\mathcal{RV}$ functions behaved like power functions (up to a factor which varies “more slowly”), $\mathcal{RPV}$ functions have a behavior close to that of exponential functions.

De Haan theory, class Π

Definition

A measurable function $f \in [a,\infty) \to \mathbb{R}$ is said to belong to the class Π if there exists a function $w : (0,\infty) \to (0,\infty)$ such that for $\lambda > 0$

$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{w(t)} = \ln \lambda;$$

we write $f \in \Pi$ or $f \in \Pi(w)$. The function w is called an *auxiliary function* for f.

The class Π of functions f is, after taking absolute values, a proper subclass of $\mathcal{SV}$.

The de Haan theory is both a direct generalization of the Karamata theory and what is needed to fill certain gaps, or boundary cases, in Karamata's main theorem.

Class Π – selected properties

- If $f \in \Pi$, then for $0 < c < d < \infty$ the relation in the definition holds uniformly for $\lambda \in [c, d]$.
- Auxiliary function is unique up to asymptotic equivalence.
- $f \in \Pi$ if and only if
$$\lim_{t\to\infty} \frac{f(\lambda t) - f(t)}{f(t) - \frac{1}{t}\int_a^t f(s)\,\mathrm{d}s} = \ln \lambda$$
for $\lambda > 0$
- $f \in \Pi$ if and only if there exists $L \in \mathcal{SV}$ such that
$$f(t) = L(t) + \int_a^t \frac{L(s)}{s}\,\mathrm{d}s.$$
- If $f \in \Pi(L)$ is integrable on finite intervals of $(0, \infty)$, then
$$L(t) \sim f(t) - \frac{1}{t}\int_a^t f(s)\,\mathrm{d}s$$
as $t \to \infty$.
- ...

De Haan theory, class Γ

Definition

A nondecreasing function $f : \mathbb{R} \to (0, \infty)$ is said to belong to the class Γ if there exists a function $v : \mathbb{R} \to (0, \infty)$ such that for all $\lambda \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{f(t + \lambda v(t))}{f(t)} = e^{\lambda};$$

we write $f \in \Gamma$ or $f \in \Gamma(v)$. The function v is called an *auxiliary function* for f.

A function is said to belong to the class $\Gamma_-(v)$ if $1/f \in \Gamma(v)$.

It holds: $\Gamma_{\pm} \subset \mathcal{RPV}(\pm\infty)$

"Inversion" of Π, but there are also other way of understanding.

Class Γ – selected properties

- If $f \in \Pi$, then the relation in the definition holds uniformly on compact λ-sets.
- $f \in \Gamma$ if and only if

$$\lim_{t\to\infty} \frac{f(t)\int_0^t\int_0^s f(\tau)\,\mathrm{d}\tau\,\mathrm{d}s}{\left(\int_0^t f(s)\,\mathrm{d}s\right)^2} = 1.$$

- If $f \in \Gamma(v)$, then

$$v(t+\lambda v(t)) \sim v(t)$$

  as $t \to \infty$ locally uniformly in $\lambda \in \mathbb{R}$ (i.e., $v \in \mathcal{SN}$; self-neglecting).
- $f \in \Gamma$ if and only if

$$f(t) = \exp\left\{\eta(t) + \int_0^t \frac{1}{\omega(s)}\,\mathrm{d}s\right\},$$

  where $\eta(t) \to c \in \mathbb{R}$ as $t \to \infty$ and $\omega \in \mathcal{SN}$. The auxiliary function of f may be taken as ω.
- ...

De Haan theory, Beurling slow variation

Definition

A measurable function $f : \mathbb{R} \to (0, \infty)$ is Beurling slowly varying if

$$\lim_{t\to\infty} \frac{f(t + \lambda f(t))}{f(t)} = 1 \quad \text{for all } \lambda \in \mathbb{R};$$

we write $f \in \mathcal{BSV}$.

Definition

If the relation holds locally uniformly in λ, then f is called self-neglecting; we write $f \in \mathcal{SN}$.

Class $\mathcal{BSV}$ – selected properties

- If $f \in \mathcal{BSV}$ is continuous, then $f \in \mathcal{SN}$.
- $f \in \mathcal{SN}$ if and only if it has the representation

$$f(t) = \varphi(t) \int_0^t \psi(s)\,\mathrm{d}s,$$

  where $\lim_{t\to\infty} \varphi(t) = 1$ and ψ is continuous with $\lim_{t\to\infty} \psi(t) = 0$.
- If $f \in \mathcal{BSV}$ is continuous, then there exists $g \in C^1$ such that $f(t) \sim g(t)$ and $g'(t) \to 0$ as $t \to \infty$.
- ...

RV and Γ

Let h be a nonnegative solution of

$$h(t)h''(t) = \varphi(t)h'^2(t),$$

where $\varphi(t) \to C \in \mathbb{R} \cup \{\pm\infty\}$ as $t \to \infty$.

- If $C \neq 1$, then $h \in \mathcal{RV}(\vartheta)$ where $\vartheta = 1/(1 - C)$; here $\vartheta = 0$ if $C = \pm\infty$.
- If $C = 1$, then $h \in \Gamma_{\pm}(|h/h'|)$, the sign $\pm$ depends on the sign of h'.

Generalized regular variation

Consider a positive $\omega \in C^1$ which satisfies $\omega'(t) > 0$ for $t \in [b, \infty)$ and $\lim_{t\to\infty} \omega(t) = \infty$.

Definition

A measurable function $f : [a, \infty) \to (0, \infty)$ is called regularly varying of index ϑ with respect to ω if $f \circ \omega^{-1} \in \mathcal{RV}(\vartheta)$; we write $f \in \mathcal{RV}_\omega(\vartheta)$. If $\vartheta = 0$, then f is called slowly varying with respect to ω; we write $f \in \mathcal{SV}_\omega$.

Many properties of generalized $\mathcal{RV}$ functions are immediate consequences of the properties of $\mathcal{RV}$ functions.
[Jaroš, Kusano 2004]

Half-linear differential equations

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \qquad \text{(HL)}$$

where $r, p \in C([a,\infty))$, $r(t) > 0$, $\Phi(u) = |u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha > 1$.
β is the conjugate number of α:

$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$

- The solution space is homogeneous but not additive.
- Bihari (1957, 1964), Elbert (1979), Mirzov (1976), ...
- If $\alpha = 2$, then (HL) reduces to a linear Sturm-Liouville equation.
- Radially symmetric solutions of the PDE with *p*-Laplacian
- Special (bordeline) case of quasilinear (generalized Emden–Fowler) DEs .
- ...

Half-linear differential equations, monotone solutions

Equation

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \qquad \text{(HL)}$$

with $p(t) > 0$ is nonoscillatory and any its nontrivial solution is eventually monotone.

- $\mathcal{IS}$ – the set of solutions y such that $y(t) > 0, y'(t) > 0$ for large t.
- $\mathcal{DS}$ – the set of solutions y such that $y(t) > 0, y'(t) < 0$ for large t.
- $\mathcal{IS}_A$ resp. $\mathcal{DS}_A$ – the sets of solutions $y \in \mathcal{IS}$ resp. $y \in \mathcal{DS}$ such that $\lim_{t\to\infty} y(t) = A$.
- $\mathcal{IS}_{A,B}$ resp. $\mathcal{DS}_{A,B}$ – the sets of solutions $y \in \mathcal{IS}$ resp. $y \in \mathcal{DS}$ such that $\lim_{t\to\infty} y(t) = A$ and $\lim_{t\to\infty} r(t)\Phi(y'(t)) = B$.

Nonoscillatory solutions – classification, existence, asymptotics, ...: Cecchi, Chanturiya, Došlá, Marini, Mirzov, ...

Solutions in the class Γ

$$(\Phi(y'))' = p(t)\Phi(y), \tag{HL}$$

where $p \in C([a,\infty))$, $p(t) > 0$, $\Phi(u) = |u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha > 1$.

Theorem

If

$$p^{-\frac{1}{\alpha}} \in \mathcal{BSV},$$

then

$$\emptyset \neq \mathcal{IS} = \mathcal{IS}_\infty \subset \Gamma\left(\left(\frac{\alpha-1}{p}\right)^{\frac{1}{\alpha}}\right).$$

Proof

- First assume $p \in C^1$. Then $(p^{-\frac{1}{\alpha}})'(t) \to 0$ as $t \to \infty$.
- Take $y \in \mathcal{IS}$. Set $w = p^{-\frac{1}{\beta}} \Phi(y'/y)$.
- w satisfies the Riccati type equation

$$\frac{w'}{p^{\frac{1}{\alpha}}(t)} = 1 - (\alpha - 1) w \left(\frac{p'(t)}{\alpha p^{\frac{\alpha+1}{\alpha}}(t)} + w^{\beta-1} \right).$$

- w satisfies $\lim_{t\to\infty} w(t) = (\alpha - 1)^{-\frac{1}{\beta}}$.
- y satisfies

$$\frac{y''(t)y(t)}{y'^2(t)} \sim 1$$

- $y \in \Gamma\left(\left(\frac{\alpha-1}{p}\right)^{\frac{1}{\alpha}}\right)$.

Proof (continuation)

- We drop the assumption on differentiability of p.
- Since $p \in \mathcal{BSV}$ there exists $\hat{p} \in C^1$ with $\hat{p}(t) \sim p(t)$ and $(\hat{p}^{-\frac{1}{\alpha}})'(t) \to 0$ as $t \to \infty$.
- For $\varepsilon \in (0, 1)$ we consider the auxiliary equations $(\Phi(u'))' = (1+\varepsilon)\hat{p}(t)\Phi(u)$ and $(\Phi(v'))' = (1-\varepsilon)\hat{p}(t)\Phi(v)$.
- For increasing solutions u, v we show

$$\left(\frac{u'(t)}{u(t)}\right)^{\alpha-1} \hat{p}^{-\frac{1}{\beta}}(t) \sim \left(\frac{1+\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}}, \ \left(\frac{v'(t)}{v(t)}\right)^{\alpha-1} \hat{p}^{-\frac{1}{\beta}}(t) \sim \left(\frac{1-\varepsilon}{\alpha-1}\right)^{\frac{1}{\beta}}$$

- From the standard result on differential inequalities, $\lim_{t\to\infty} \left(\frac{y'(t)}{y(t)}\right)^{\alpha-1} p^{-\frac{1}{\beta}}(t) = (\alpha-1)^{-\frac{1}{\beta}}$.
- The rest of the proof is the same as in the previous part.

Remarks

- Half-linear extension of [Omey 1997]. (The Wronskian identity is not at disposal, ...)
- By-product: $\mathcal{IS} \subset \mathcal{RPV}(\infty)$ (Open problem posed by Kusano at CDDE conference in 2000).

Remarks

- (Decreasing solutions) If $\lim_{t\to\infty}\int_a^t\left(\int_s^t p(z)\,dz\right)^{\beta-1} ds=\infty$ and $p^{-\frac{1}{\alpha}}\in\mathcal{BSV}$, then

$$\emptyset\neq\mathcal{DS}=\mathcal{DS}_0\subset\Gamma_-\left(((\alpha-1)/p)^{\frac{1}{\alpha}}\right).$$

  (The reduction of order formula is not at disposal for HL equations.) The proof require a different approach. The concept of principal solutions finds an application. The results are new even in the linear case.

- Using similar ideas we can prove: If $(\tilde{p}^{-\frac{1}{\alpha}}(t))'\to A\neq 0$ with $\tilde{p}\sim p$, then $\mathcal{IS}\subset\mathcal{NRV}(\vartheta_1)$, $\mathcal{DS}\subset\mathcal{NRV}(\vartheta_2)$, $\vartheta_i=\vartheta_i(A)$.
  Cf. Jaroš, Kusano, Tanigawa, 2003: $t^{\alpha-1}\int_t^\infty p(s)\,ds\to B\Rightarrow$ $\exists y_i\in\mathcal{NRV}(\vartheta_i)$, $\vartheta_i=\vartheta_i(B)$, $i=1,2$ (by a fixed point approach).

there exist solutions . . . vs. all solutions . . .

Remarks

- By-product: There are solutions y_1, y_2 such that

$$y_i'(t) \sim \pm \left(\frac{p(t)}{\alpha - 1} \right)^{\frac{1}{\alpha}} y_i(t).$$

  Half-linear extension of Hartman's-Wintner's result.

- A connection with generalized regular variation: For example, if $p^{-\frac{1}{\alpha}} \in \mathcal{BSV}$, then $\mathcal{IS} \subset \mathcal{NRV}_\omega((\alpha - 1)^{-\frac{1}{\alpha}})$, where $\omega(t) = \left\{ \int_a^t p^{\frac{1}{\alpha}}(s)\, ds \right\}$.

- The results can be extended to the more general equation $(r(t)\Phi(y'))' = p(t)\Phi(y)$, but the extension is not immediate. Some steps in the proof require a different approach. Among others, uniformity of self-neglecting functions plays an important role. The results are new even in the linear case.

Solutions in the class Π, $(\Phi(y'))' = p(t)\Phi(y)$

Theorem

Let $p \in \mathcal{RV}(-\alpha)$. If $L_p(t) \to 0$ as $t \to \infty$, where L_p is $\mathcal{SV}$ component of p, then $\mathcal{DS} \subset \mathcal{NSV}$. For $y \in \mathcal{DS}$, one has $-y \in \Pi(-ty'(t))$. Moreover,
(i) If $\int_a^\infty (sp(s))^{\frac{1}{\alpha-1}}\,\mathrm{d}s = \infty$, then

$$y(t) = \exp\left\{-\int_a^t \left(\frac{sp(s)}{\alpha-1}\right)^{\frac{1}{\alpha-1}} (1+o(1))\,\mathrm{d}s\right\}$$

as $t \to \infty$, and $y \in \mathcal{DS}_{0,0}$.
(ii) If $\int_a^\infty (sp(s))^{\frac{1}{\alpha-1}}\,\mathrm{d}s < \infty$, then

$$y(t) = y(\infty)\exp\left\{\int_t^\infty \left(\frac{sp(s)}{\alpha-1}\right)^{\frac{1}{\alpha-1}} (1+o(1))\,\mathrm{d}s\right\}$$

as $t \to \infty$, and $y \in \mathcal{DS}_{B,0}$, $|y(\infty) - y(t)| \in \mathcal{SV}$, $L_p^{\beta-1}(t)/(y(\infty) - y(t)) = o(1)$.

Proof

- Take $y \in \mathcal{DS}$.
- We show that $ty'(t)/y(t) \to 0$ as $t \to \infty$, which implies $y \in \mathcal{NSV}$.
- From $-\Phi(y') \in \mathcal{RV}(1-\alpha)$, we get $-y(t) \in \Pi(-ty'(t))$
- Set $h(t) = t^{\alpha-1}\Phi(y'(t)) - (\alpha-1)\int_a^t s^{\alpha-2}\Phi(y'(s))\,ds$.
- Then $h \in \Pi(-(\alpha-1)t^{\alpha-1}\Phi(y'(t)))$ and $h \in \Pi(th'(t))$.
- From the uniqueness of the auxiliary function up to asymptotic equivalence and $h'(t) = t^{\alpha-1}p(t)\Phi(y(t))$, we get $\frac{y'(t)}{y(t)} = -(1+o(1))\tilde{p}(t)$, where $\tilde{p}(t) = \left(\frac{tp(t)}{\alpha-1}\right)^{\frac{1}{\alpha-1}}$ as $t \to \infty$.
- We distinguish the cases $\int^\infty \tilde{p}(s)\,\mathrm{d}s = \infty$ and $\int^\infty \tilde{p}(s)\,\mathrm{d}s < \infty$. Then we integrate the asymptotic relation to obtain formulas. Moreover, we find $y(t) \to 0$ resp. $y(t) \to y(\infty) \in (0, \infty)$.

Remarks

- Half-linear extension of [Geluk 1990].
- An alternative proof can be found (the Karamata integration theorem)
- To prove $\mathcal{DS} \subset \mathcal{NSV}$ it is sufficient to assume the integral condition $t^{\alpha-1} \int_t^\infty p(s)\, ds \to 0$. (Half-linear extension of [Marić, Tomić 1990].)
- The necessity can be shown.

Remarks

- The results can be extended to the more general equation $(r(t)\Phi(y'))' = p(t)\Phi(y)$.
  - We assume $p(t) \in \mathcal{RV}(\delta)$ and $r(t) \in \mathcal{RV}(\delta + \alpha)$ (the relation between the indices is natural), and $L_p(t)/L_r(t) \to 0$, where L_p and L_r are $\mathcal{SV}$ components of p and r, respectively.
  - If $\delta < -1$, then we proceed similarly as above.
  - If $\delta > -1$, then $\mathcal{SV}$ solutions must be sought among $\mathcal{IS}$ solutions! In particular, $\mathcal{IS} \subset \mathcal{NSV}$ and asymptotic formulas are established.
  - The results are new even in the linear case.
  - The case $\delta = -1$ is substantially different — we will discuss it later.

Non-$\mathcal{SV}$ solutions of $(r(t)\Phi(y'))' = p(t)\Phi(y)$

The reduction of order formula or the transformation of dependent variable are not at disposal! The Karamata theorem cannot directly be used (since we are in the "critical" case).
The technique of the proof:

- Either "ab ovo" (Riccati, de Haan theory, ...)
- Or, by the reciprocity principle, with the use of the previous results on $\mathcal{SV}$ solutions and the Karamata theory.

It is shown that the "remaining" solutions are in

$$\mathcal{RV}\left(\frac{\delta+1}{1-\alpha}\right)$$

and satisfy certain formulae.

Classification

The above results on $\mathcal{SV}$ and non-$\mathcal{SV}$ solutions, along with some further observations lead to the following classification of nonoscillatory solutions in the framework of regular variation.

Notation:

$$\mathcal{S}_{(\mathcal{N})\mathcal{SV}} = \mathcal{S} \cap (\mathcal{N})\mathcal{SV} \text{ and } \mathcal{S}_{(\mathcal{N})\mathcal{RV}} = \mathcal{S} \cap (\mathcal{N})\mathcal{RV}\left(\frac{\delta+1}{1-\alpha}\right),$$

$$J = \int_a^\infty \frac{1}{t}\left(\frac{L_p(t)}{L_r(t)}\right)^{\beta-1} \mathrm{d}t \text{ and } R = \int_a^\infty \frac{1}{t} \cdot \frac{L_p(t)}{L_r(t)}\,\mathrm{d}t,$$

where L_p and L_r are $\mathcal{SV}$ components of p and r, respectively.

Remark: In general, $J \neq R$. If $\alpha = 2$, then $J = R$.

Classification

Let $p \in \mathcal{RV}(\delta)$, $r \in \mathcal{RV}(\delta + \alpha)$, and $L_p(t)/L_r(t) \to 0$ as $t \to 0$.
Assume that $\delta < -1$.

- If $J = \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{0,0}$.
- If $J < \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}} = \mathcal{DS} = \mathcal{DS}_{B,0}$.
- If $R = \infty$, then $\mathcal{S}_{\mathcal{RV}} = \mathcal{S}_{\mathcal{NRV}} = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}$.
- If $R < \infty$, then $\mathcal{S}_{\mathcal{RV}} = \mathcal{S}_{\mathcal{NRV}} = \mathcal{IS} = \mathcal{IS}_{\infty,B}$.

Assume that $\delta > -1$.

- If $J = \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{\infty,\infty}$.
- If $J < \infty$, then $\mathcal{S}_{\mathcal{SV}} = \mathcal{S}_{\mathcal{NSV}} = \mathcal{IS} = \mathcal{IS}_{B,\infty}$.
- If $R = \infty$, then $\mathcal{S}_{\mathcal{RV}} = \mathcal{S}_{\mathcal{NRV}} = \mathcal{DS} = \mathcal{DS}_{0,0}$.
- If $R < \infty$, then $\mathcal{S}_{\mathcal{RV}} = \mathcal{S}_{\mathcal{NRV}} = \mathcal{DS} = \mathcal{DS}_{0,B}$.

In all the cases, for any $y \in \mathcal{DS}$ and any $y \in \mathcal{IS}$ asymptotic formulae are established.
The conditions (naturally) match existing general existence conditions for nonemptiness in the relevant subclasses of $\mathcal{DS}$ and $\mathcal{IS}$.

The case $t^\alpha p(t)/r(t) \to C > 0$

The condition $L_p(t)/L_r(t) \to 0$ can be seen as $t^\alpha p(t)/r(t) \to 0$. This suggests to study $t^\alpha p(t)/r(t) \to C > 0$ as $t \to \infty$.

In the linear case: Either "ab ovo" or by the (linear) transformation of dependent variable "$y = hu$", with the use of the previous result (we set $h(t) = t^\vartheta$ with a suitable ϑ, and then u is a $\mathcal{SV}$ solution of the auxiliary equation).

This technique is not at disposal in the HL case. However, we can try to apply a transformation into a modified generalized Riccati equation which can partially substitute it. In fact, this leads to an "asymptotic linearization".

Example of formula

$$y(t) = t^\vartheta \exp\left\{\int_a^t (1 + o(1))B\frac{L(s)}{s}\,\mathrm{d}s\right\} \quad \text{as } t \to \infty,$$

where $L(t) = t^\alpha p(t)/r(t) - C - \Phi(\vartheta)\left(\gamma - tr'(t)/r(t)\right)$, B is a certain constant, γ is index of $\mathcal{RV}$ of r, ϑ is a root of a certain associated algebraic equation.

A generalization

The previous results can be generalized via a transformation of independent variable, where we distinguish the cases $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s = \infty$ and $\int_a^\infty r^{1-\beta}(s)\,\mathrm{d}s < \infty$. This generalization enables us to include, in particular, the following:

- The border-line case $\delta = -1$: The "double-root case" which can be somehow associated to (transformed) perturbations of Euler type or other equations.
- Some equations with non-$\mathcal{RV}$ coefficients.

Half-linear differential equations — asymptotics

- Generalization of results for linear DE's; many of the observations are new in the linear case
- A refinement on information about behavior of solutions in standard asymptotic classes
- Regular or rapid variation of all positive solutions
- A refinement on information about behavior of regularly and rapidly varying solutions
- New (combinations of) methods in half-linear asymptotic theory

Future research directions, open problems

- Improving the basic conditions in the sense of integral ones, e.g., in terms of $\lim_{t\to\infty}\left(\int^t r^{1-\beta}\right)^{\alpha-1}\int_t^\infty p$.
- The opposite sign condition or no sign condition for p.
  As for the opposite sign case:
  - Some of the above ideas work.
  - Substantially different classification of nonoscillatory solutions.
  - Problem with showing regular variation of ALL (positive) solutions. The existence is OK (by a fixed point theorem).

Future research directions, open problems

- To describe the behavior of solutions in a more precise way: "second order" behavior, asymptotic formulas or estimates for remainders, ...
- Difference equations. The discrete Karamata theory is at disposal.
- Differential equations with deviated arguments. Existing asymptotic theory for first order functional differential equations could be helpful.
- ...

Future research directions, open problems

Nearly (half-)linear equation

$$(r(t)G(y'))' = p(t)F(y),$$

where $uF(u) > 0$, $uG(u) > 0$ for $u \neq 0$, $F(|\cdot|), G(|\cdot|)$ are $\mathcal{RV}$ or $\mathcal{RV}_0$ functions with the SAME positive indices.

- "Neither superlinear, nor sublinear."
- Examples of $F(u), G(u)$: $L(u)\Phi_\alpha(u)$ with $L(u) \sim \mathrm{const}$, $\Phi_\alpha(u)\Phi_\gamma(A + B\Phi_\delta(u))$ [spec. $\frac{u}{\sqrt{1\pm u^2}}$], $\Phi_\alpha(u)|\ln|\ln|u|||^\gamma|\ln|u||^\delta$, ...
- It turns out that a modification of some methods known from the (half-)linear theory is a useful tool rather than the methods known from the theory of general nonlinear (superlinear or sublinear) equations.
- Some phenomena may occur for nearly (half-)linear equations which cannot happen in the purely (half-)linear case.

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