Dirichlet problem with impulses at state-dependent moments

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Vector case with p barriers given explicitely $t = \gamma_i(\mathbf{x})$

$$a < \gamma_1(\mathbf{x}) < \gamma_2(\mathbf{x}) < \cdots < \gamma_p(\mathbf{x}) < b,$$

$$\mathbf{x} \in D \subset \mathbb{R}^n, \ n, p \in \mathbb{N}, \quad \gamma_i \in \mathbb{C}(D; \mathbb{R}), \ i = 1, \dots, p.$$

$$\mathbf{z}'(t) = \mathbf{f}(t, \mathbf{z}(t)) \text{ for a.e. } t \in [a, b], \tag{1}$$

$$\mathbf{z}(t+) - \mathbf{z}(t) = \mathbf{J}_i(t, \mathbf{z}(t)) \text{ for } t \text{ such that } t = \gamma_i(\mathbf{z}(t)), \tag{2}$$

$$\ell(\mathbf{z}) = \mathbf{c}_0, \quad \mathbf{c}_0 \in \mathbb{R}^n. \tag{3}$$

We assume that

$$\mathbf{f} \in Car([a, b] \times \mathbb{R}^n; \mathbb{R}^n), \quad \mathbf{J}_i \in \mathbb{C}([a, b] \times \mathbb{R}^n; \mathbb{R}^n),$$

$\ell : \mathbb{G}_L([a, b]; \mathbb{R}^n) \to \mathbb{R}^n$ is linear bounded.

State-dependent impulses

Definition

$\mathbf{z} \in \mathbb{G}_L([a,b];\mathbb{R}^n)$ is a solution of problem (1)–(3), if

- $\mathbf{z}$ satisfies equation (1) for a.e. $t \in [a,b]$,
- $\mathbf{z}$ fulfils conditions (2) and (3).

We prove the existence of a solution $\mathbf{z}$ of problem (1)–(3) having the following properties:

- for each $i \in \{1,\dots,p\}$ there exists a unique $\tau_i \in (a,b)$ such that $\gamma_i(\mathbf{z}(\tau_i)) = \tau_i$,
- $a = \tau_0 < \tau_1 < \dots < \tau_p < \tau_{p+1} = b$,
- the restrictions $\mathbf{z}|_{[\tau_0,\tau_1]}$ and $\mathbf{z}|_{(\tau_i,\tau_{i+1}]}$, $i = 1,\dots,p$, are absolutely continuous.

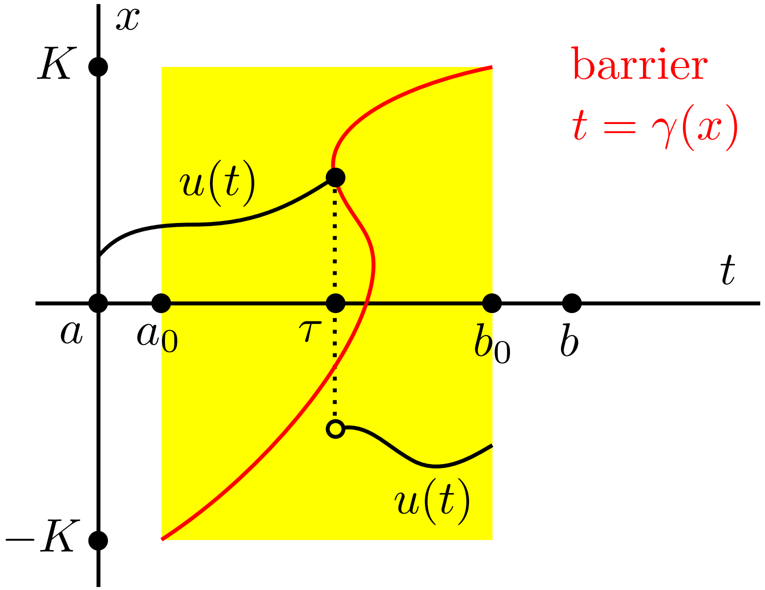
Analytical-topological approach based on the papers

- **Rachůnková, I., Tomeček, J.**, A new approach to BVPs with state-dependent impulses, *Boundary Value Problems 2013*, **2013:22**, 1–13.
- **Rachůnková, I., Tomeček, J.**, Second order BVPs with state-dependent impulses via lower and upper functions, *Central European Journ. Math.* **12** (2014), 128–140.
- **I. Rachůnková, J. Tomeček**, Impulsive system of ODEs with general linear boundary conditions. *E. J. Qualitative Theory of Diff. Equ.*, 25 (2013), 1–16.
- **I. Rachůnková, J. Tomeček**, Existence principle for higher order nonlinear differential equations with state-dependent impulses via fixed point theorem, *Boundary Value Problems 2014*, **2014:118**, 1–15.

- **I. Rachůnková, J. Tomeček**, Existence principle for BVPs with state-dependent impulses, *Topol. Methods Nonlinear Anal.*, **44** (2) (2014), 349–368.

- **I. Rachůnková, J. Tomeček**, Fixed point problem associated with state-dependent impulsive boundary value problems, *Boundary Value Problems 2014*, **2014:172**, 1–17.

- **Rachůnková, I., Tomeček, J.**, State-Dependent Impulses. Boundary Value Problems on Compact Interval, *Atlantis Press, Springer*, 2015.


x
K
barrier
t = γ(x)
u(t)
t
a
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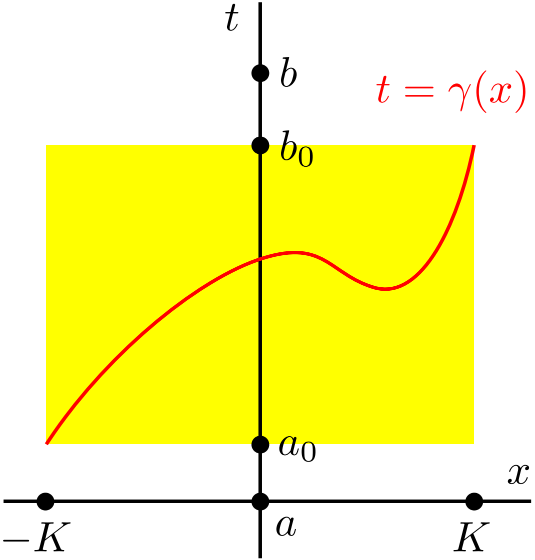
Barrier $t = \gamma(x)$

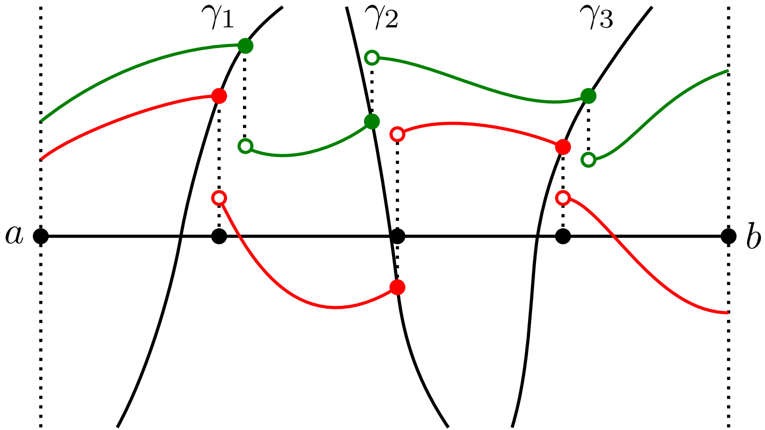


1. The space where we search solutions

- $\ell : \mathbb{G}_L([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is linear bounded.
- $\mathbb{G}_L([a,b];\mathbb{R}^n)$ is a Banach space of left-continuous regulated mappings.
- A mapping $\mathbf{z} : [a,b] \to \mathbb{R}^n$ is left-continuous regulated on $[a,b]$ if for each $t \in (a,b]$ and each $s \in [a,b)$

$$\lim_{\xi \to t-} \mathbf{z}(\xi) = \mathbf{z}(t) = \mathbf{z}(t-) \in \mathbb{R}^n, \quad \lim_{\xi \to s+} \mathbf{z}(\xi) = \mathbf{z}(s+) \in \mathbb{R}^n.$$

Two solutions of impulsive BVP have jumps at different points



Main differences between FIX and S–D impulses

2. Number of intersection points

There are barriers γ and solutions u of differential equations such that the equation

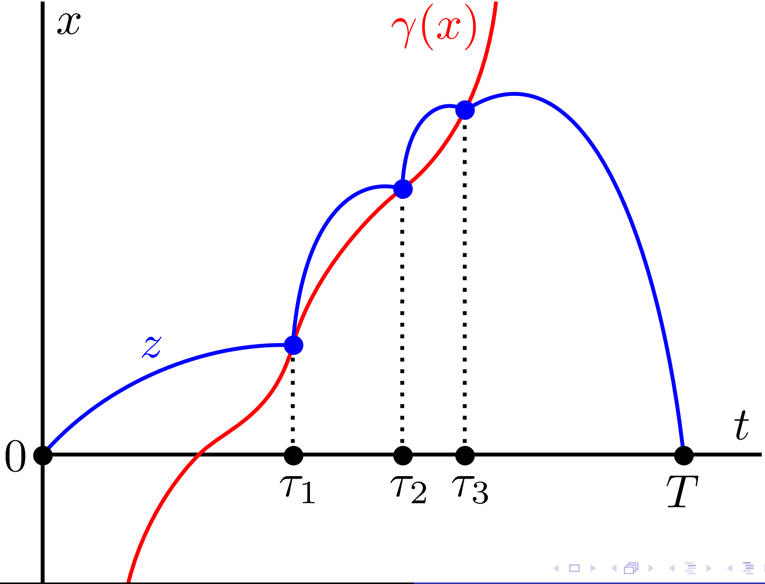
$$\tau = \gamma(u(\tau))$$

has more than one solution τ_u. In this case the solution u has more intersection points with the barrier γ. Then the mapping

$$\mathcal{P} : u \mapsto \tau_u$$

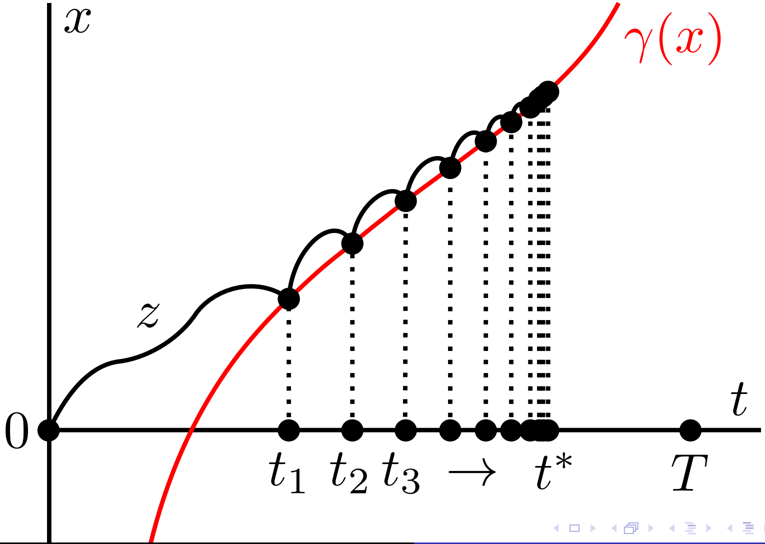
is a multivalued mapping. This makes a transformation of problem (1)–(3) to an operator equation difficult.

Solution z of the impulsive Dirichlet problem has three intersection points with the barrier γ



Solution z of an impulsive differential equation has infinitely many intersection points with the barrier γ

Impulsive BVP cannot be solved



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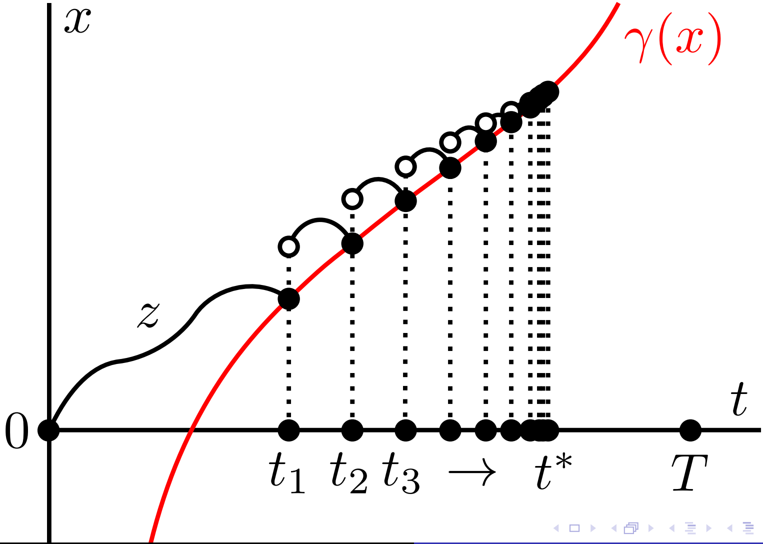


Consider a solution u of the initial problem

$$u''(t) = 0, \quad u(0) = -0.9, \quad u'(0) = 0,$$

with the imupulse condition

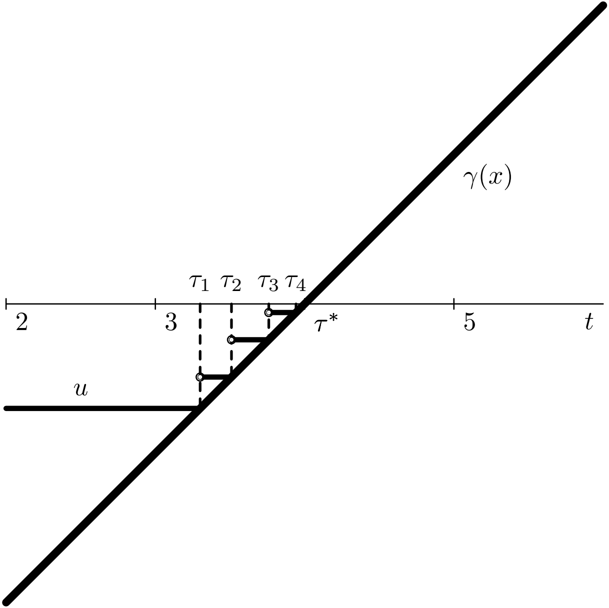
$$u(\tau+) - u(\tau) = J(u(\tau)), \quad \tau = \gamma(u(\tau)).$$

Here

$$J(x) = -x^2 - x, \quad \gamma(x) = x + 4, \quad \text{for } x \in [-3, 3].$$

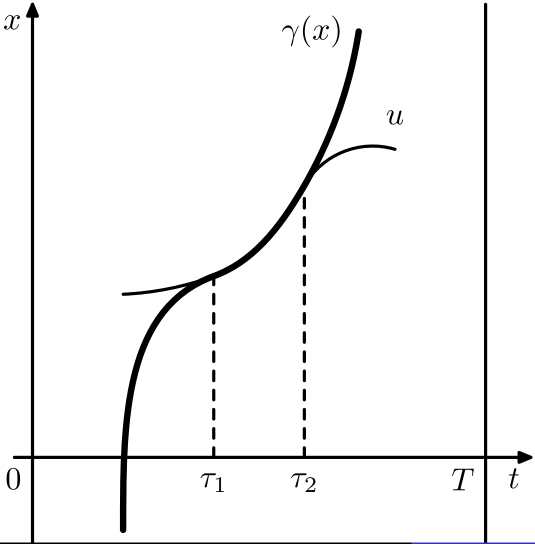
The solution u is subject to an impulse effect at infinitely many moments τ_n, and $\lim_{n\to\infty} \tau_n = \tau^* = 4$, $\lim_{n\to\infty} u(\tau_n) = 0$. Such solution cannot be extended to $T > 4$.

Beating of solution u



Solution u of a differential equation is pasted together with the barrier γ

Impulsive equation (and impulsive BVP) cannot be solved



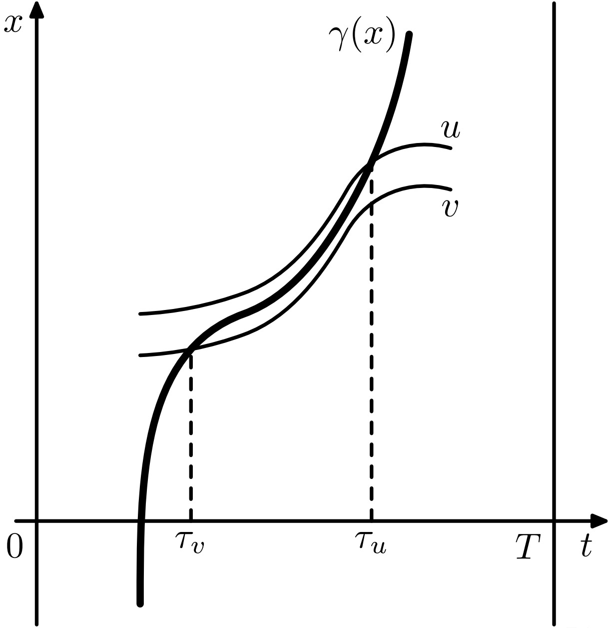
3. Intersection point τ_u need not depend on u continuously

Consider functions in $\mathcal{C}[0, T]$ having just one intersection point with γ. The next figure shows functions u and v which are close to each other while their intersection points τ_u and τ_v are not. In this case the functional

$$\mathcal{P} : u \mapsto \tau_u$$

can be defined on the set of such functions, but $\mathcal{P}$ is not continuous. This makes a transformation of problem (1)–(3) to an operator equation difficult.

Point τ_u does not depend on u continuously



4. Resonance

$$\mathbf{z}'(t) = \mathbf{f}(t, \mathbf{z}(t)) \text{ for a.e. } t \in [a, b], \tag{4}$$

$$\mathbf{z}(t+) - \mathbf{z}(t) = \mathbf{J}_i(t, \mathbf{z}(t)) \text{ for } t \text{ such that } t = \gamma_i(\mathbf{z}(t)), \tag{5}$$

$$\ell(\mathbf{z}) = \mathbf{c}_0, \quad \mathbf{c}_0 \in \mathbb{R}^n, \tag{6}$$

where

$$\mathbf{f}(t, \mathbf{z}(t)) = A(t)\mathbf{z}(t) + \mathbf{h}(t, \mathbf{z}(t)),$$

$$\mathbf{J}_i(t, \mathbf{z}(t)) = B_i\mathbf{z}(t) + \mathbf{m}_i(t, \mathbf{z}(t)), \; i = 1, \ldots, p.$$

The linear homogeneous problem corresponding to problem (4)-(6)

$$\mathbf{z}'(t) = A(t)\mathbf{z}(t), \quad \ell(\mathbf{z}) = \mathbf{0}. \tag{7}$$

5. Fredholm property

We have the Dirichlet BVP with one state–dependent impulse

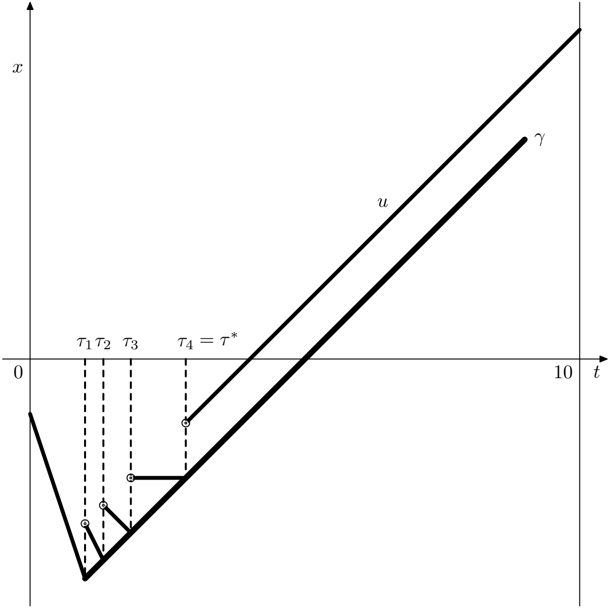
$$(7)\begin{cases} u''(t) = 0, \quad u(0) = -1, \quad u(10) = 0, \\ u(\tau+) - u(\tau) = 1, \quad u'(\tau+) - u'(\tau-) = 1, \\ \tau = 5 + u(\tau) \quad \text{for } \tau \in [1, 9]. \end{cases}$$

and the same BVP with one impulse at fixed point

$$(8)\begin{cases} u''(t) = 0, \quad u(0) = -1, \quad u(10) = 0, \\ u(t_0+) - u(t_0) = 1, \quad u'(t_0+) - u'(t_0-) = 1, \\ t_0 \in [1, 9] \quad \text{is fixed}. \end{cases}$$

Since the problem $u''(t) = 0$, $u(0) = u(10) = 0$ has only the trivial solution and so the Green function exists, problem (8) is solvable. But problem (7) is not solvable!

Solution u of $u'' = 0, u(0) = -1, u'(0) = -3$



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We assume that

$$\mathbf{f} \in Car([a, b] \times \mathbb{R}^n; \mathbb{R}^n), \quad \mathbf{J}_i \in \mathbb{C}([a, b] \times \mathbb{R}^n; \mathbb{R}^n),$$

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- the restrictions $\mathbf{z}|_{[\tau_0,\tau_1]}$ and $\mathbf{z}|_{(\tau_i,\tau_{i+1}]}$, $i = 1,\dots,p$, are absolutely continuous.

Representation of a linear bounded operator

- **M.Tvrdý**: Linear integral equations in the space of regulated functions, *Mathematica Bohemica* 123 (1998), 177–212.

$\ell : \mathbb{G}_L([a,b];\mathbb{R}^n) \to \mathbb{R}^n$ is a linear bounded operator if and only if there exist $K \in \mathbb{R}^{n\times n}$ and $V \in \mathbb{BV}([a,b];\mathbb{R}^{n\times n})$ such that

$$\ell(\mathbf{z}) = K\mathbf{z}(a) + \int_a^b V(t)\,\mathrm{d}[\mathbf{z}(t)], \quad \mathbf{z} \in \mathbb{G}_L([a,b];\mathbb{R}^n), \qquad (8)$$

where the integral in (8) is the Kurzweil-Stieltjes integral.

Green's matrix

If det $K \neq 0$, then there exists the Green's matrix G of the corresponding linear homogeneous problem

$$\mathbf{z}'(t) = \mathbf{0}, \quad \ell(\mathbf{z}) = \mathbf{0}. \tag{5}$$

The matrix G takes the form

$$G(t,\tau) = \begin{cases} G_1(t,\tau), & a \le t \le \tau \le b, \\ G_2(t,\tau), & a \le \tau < t \le b, \end{cases}$$

where

$$G_1(t,\tau) = -K^{-1}V(\tau), \quad G_2(t,\tau) = -K^{-1}V(\tau) + I, \quad t,\tau \in [a,b].$$

First operator representation of problem (1)-(3)

$\mathcal{F} : \mathbb{G}_L([a,b];\mathbb{R}^n) \to \mathbb{G}_L([a,b];\mathbb{R}^n)$

$$(\mathcal{F}\mathbf{z})(t) = \int_a^b G(t,s)\mathbf{f}(s,\mathbf{z}(s))\,\mathrm{d}s + \sum_{i=1}^{p} G(t,\tau_i)\mathbf{J}_i(\tau_i,\mathbf{z}(\tau_i))$$

$$+Y(t)\,[\ell(Y)]^{-1}\,\mathbf{c},$$

τ_i depends on $\mathbf{z}$ through $\tau_i = \gamma_i(\mathbf{z}(\tau_i))$, $i = 1,\dots,p$.

$$\mathcal{P}_i : \mathbf{z} \to \tau_i, \quad i = 1,\dots,p.$$

$\mathbf{z}$ is a fixed point of operator $\mathcal{F}$ iff $\mathbf{z}$ is a solution of problem (1)-(3). $\mathcal{P}_i$ can be multivalued mapping and need not be continuous.

Even if all mappings $\mathcal{P}_i$ are absolutely continuous functionals, the operator

$$\mathcal{F} : \mathbb{G}_L([a,b];\mathbb{R}^n) \to \mathbb{G}_L([a,b];\mathbb{R}^n)$$

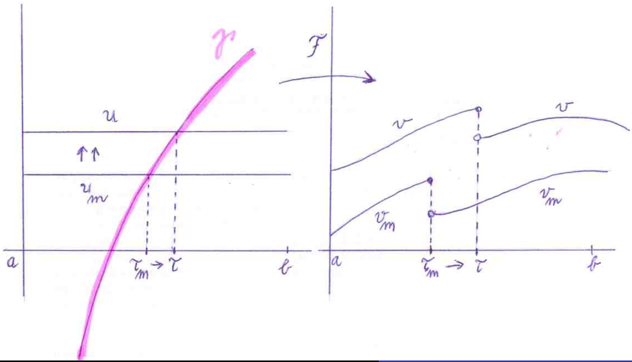
$$(\mathcal{F}\mathbf{z})(t) = \int_a^b G(t,s)\mathbf{f}(s,\mathbf{z}(s))\,\mathrm{d}s + \sum_{i=1}^{p} G(t,\mathcal{P}_i(\mathbf{z}))\mathbf{J}_i(\mathcal{P}_i(\mathbf{z}),\mathbf{z}(\mathcal{P}_i(\mathbf{z})))$$

$$+Y(t)\,[\ell(Y)]^{-1}\,\mathbf{c},$$

is not continuous.

$\mathcal{F}$ is not continuous

$$u(t) = 1, \quad u_m(t) = 1 - 1/m, \qquad t \in [a,b], \; m \in \mathbb{N},$$
$$v = \mathcal{F}u, \quad v_m = \mathcal{F}u_m, \quad m \in \mathbb{N},$$
$$\lim_{m\to\infty} u_m(t) = u(t) \quad \text{uniformly on} \;\; [a,b],$$
$$\lim_{m\to\infty} v_m(\tau) = v(\tau+) \neq v(\tau).$$



Transversality conditions

Consider $\mu_j \in (0, \infty)$, $j = 1, \dots, n$, and denote $\mathbf{x} = (x_1, \dots, x_n)^T$, $\mathbf{y} = (y_1, \dots, y_n)^T$, $\mathbf{u} = (u_1, \dots, u_n)^T$,
$A = \{\mathbf{x} \in \mathbb{R}^n : |x_j| \le \mu_j,\ j = 1, \dots, n\}$.
Assume:

- $\exists$ disjoint subintervals $[a_i, b_i]$ of $(a, b) : a_1 < \dots < a_p$, $a_i \le \gamma_i(\mathbf{x}) \le b_i,\ i = 1, \dots, p,\ \mathbf{x} \in A$,
- $\forall i = 1, \dots, p,\ j = 1, \dots, n,\ \exists \lambda_{ij} \in [0, \infty)$: $|\gamma_i(\mathbf{x}) - \gamma_i(\mathbf{y})| \le \sum_{j=1}^n \lambda_{ij} |x_j - y_j|$, $\mathbf{x}, \mathbf{y} \in A$.
- $\rho_j \in (0, \infty)$, $j = 1, \dots, n$, satisfy

$$\sum_{j=1}^n \lambda_{ij} \rho_j < 1 \quad \text{for } i = 1, \dots, p.$$

- $$B = \{\mathbf{u} \in \mathbb{W}^{1,\infty}([a, b]; \mathbb{R}^n) : \|u_j\|_\infty < \mu_j,\ \|u_j'\|_\infty < \rho_j,\ j = 1, \dots, n\}$$

Lemma

For each $\mathbf{u} \in \overline{B}$ and $i \in \{1, \dots, p\}$ there exists a unique root $t = \tau_i \in (a, b)$ of the function

$$\sigma(t) = \gamma_i(\mathbf{u}(t)) - t.$$

We define a continuous functional $\mathcal{P}_i : \overline{B} \to (a, b)$ by

$$\mathcal{P}_i \mathbf{u} = \tau_i, \quad \mathbf{u} \in \overline{B}, \ i = 1, \dots, p,$$

and the set $\Omega = B^{p+1} \subset \mathbb{X}$, where

$$\mathbb{X} = \left(\mathbb{W}^{1,\infty}([a, b]; \mathbb{R}^n)\right)^{p+1},$$

is the Sobolev space equipped with the norm

$$\|U\|_{\mathbb{X}} = \sum_{k=1}^{p+1} \|\mathbf{u}_k\|_{1,\infty} \quad \text{for } U = (\mathbf{u}_1, \dots, \mathbf{u}_{p+1}) \in \mathbb{X}.$$

Now, assume:

$$\det K \neq 0, \ \exists \tilde{f} \in \mathbb{R} : |\mathbf{f}(t,\mathbf{x})| \leq \tilde{f}, \text{a.e. } t \in [a,b], \text{ all } \mathbf{x} \in \mathbb{R}^n,$$

and consider the operator $\mathcal{F} : \overline{\Omega} \to \mathbb{X}$,

$$(\mathcal{F}U)_k(t) = \int_a^b G(t,s) \sum_{i=1}^{p+1} \chi_{(\tau_{i-1},\tau_i)}(s)\mathbf{f}(s,\mathbf{u}_i(s))\,\mathrm{d}s + \sum_{i=k}^{p} G_1(t,\tau_i)\mathbf{J}_i(\tau_i,\mathbf{u}_i(\tau_i)) + \sum_{i=1}^{k-1} G_2(t,\tau_i)\mathbf{J}_i(\tau_i,\mathbf{u}_i(\tau_i)) + Y(t)\,[\ell(Y)]^{-1}\,\mathbf{c},$$

where $U = (\mathbf{u}_1, \ldots, \mathbf{u}_{p+1})$, $k = 1, \ldots, p+1$, $\tau_0 = a$, $\tau_{p+1} = b$, and τ_i depends on $\mathbf{u}_i$ through $\tau_i = \gamma_i(\mathbf{u}_i(\tau_i))$, $i = 1, \ldots, p$.

The operator $\mathcal{F}$ is not compact on $\overline{\Omega} \subset \mathbb{X}$. The problem lies with the chosen Banach space $\mathbb{X} = \left(\mathbb{W}^{1,\infty}([a,b];\mathbb{R}^n)\right)^{p+1}$. Therefore we define the operator $\mathcal{G} : \overline{\Omega} \to \mathbb{X}$,

$$(\mathcal{G}U)_k(t) = \begin{cases} (\mathcal{F}U)_k(\tau_{k-1}) + \int_{\tau_{k-1}}^{t} f(s, \mathbf{u}_k(s))\,\mathrm{d}s & \text{for } t < \tau_{k-1}, \\ (\mathcal{F}U)_k(t) & \text{for } \tau_{k-1} \le t \le \tau_k, \\ (\mathcal{F}U)_k(\tau_k) + \int_{\tau_k}^{t} f(s, \mathbf{u}_k(s))\,\mathrm{d}s & \text{for } t > \tau_k, \end{cases}$$

where $t \in [a,b]$, $U = (\mathbf{u}_1, \ldots, \mathbf{u}_{p+1})$, $k = 1, \ldots, p+1$, $\tau_0 = a$, $\tau_{p+1} = b$, and τ_i depends on $\mathbf{u}_i$ through $\tau_i = \gamma_i(\mathbf{u}_i(\tau_i))$, $i = 1, \ldots, p$.

Under some additional assumptions we have proved that the operator $\mathcal{G}$ is compact on $\overline{\Omega} \subset \mathbb{X}$.

Theorem 1

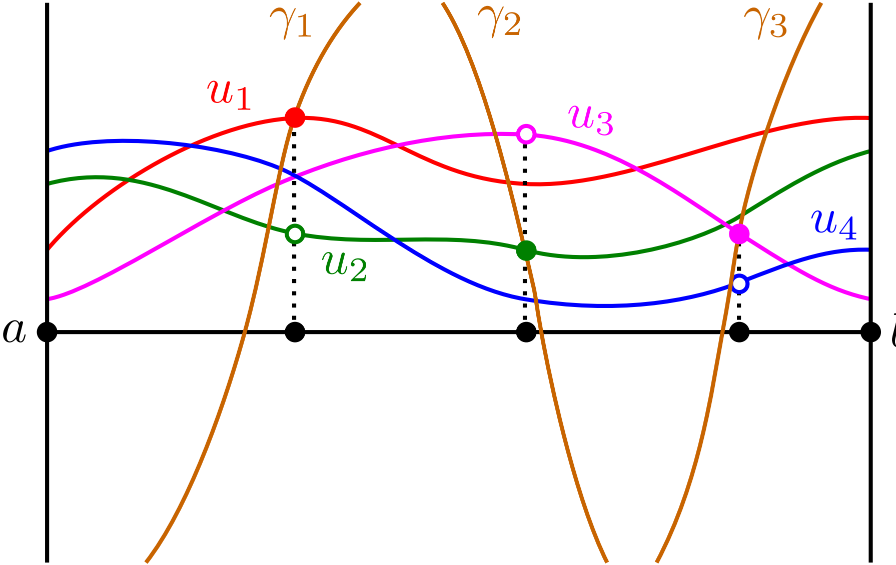
Let the following conditions be satisfied:

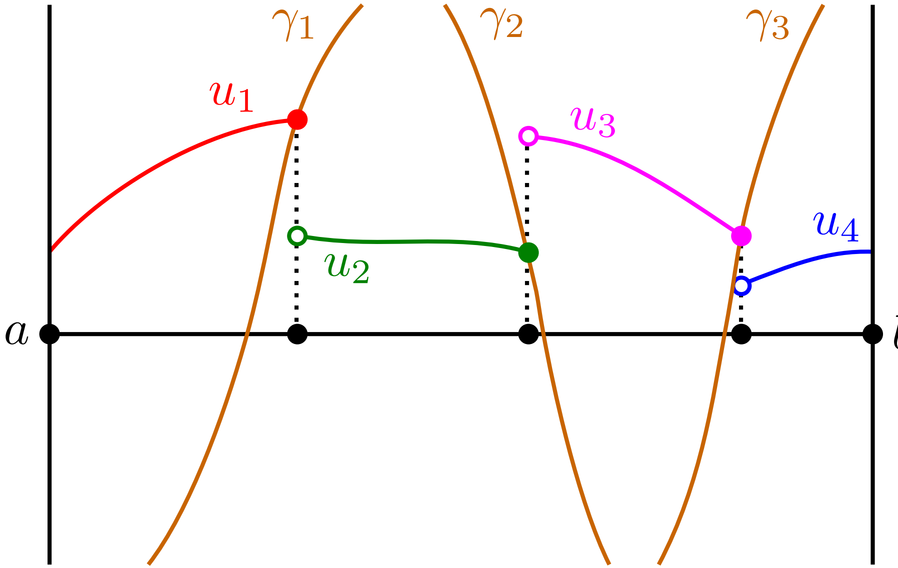
- *transversality conditions*
- $\det K \neq 0$,
- $\exists \tilde{f} \in \mathbb{R} : |\mathbf{f}(t,\mathbf{x})| \leq \tilde{f}$, *a.e.* $t \in [a,b]$, *all* $\mathbf{x} \in \mathbb{R}^n$,
- $\gamma_i(\mathbf{x}+\mathbf{J}_i(t,\mathbf{x})) \leq \gamma_i(\mathbf{x})$ *for all* $(t,\mathbf{x}) \in [a,b] \times A$, $i = 1,\ldots,p$.

If $U = (\mathbf{u}_1,\ldots,\mathbf{u}_{p+1})$ *is a fixed point of the operator* $\mathcal{G}$, *then the function*

$$\mathbf{z}(t) = \begin{cases} \mathbf{u}_1(t), & t \in [a,\tau_1], \\ \mathbf{u}_2(t), & t \in (\tau_1,\tau_2], \\ \ldots & \ldots \\ \mathbf{u}_{p+1}(t), & t \in (\tau_p, b]. \end{cases}$$

is a solution of problem (1)–(3).


γ_1
γ_2
γ_3
u_1
u_2
u_3
u_4
a


γ_1
γ_2
γ_3
u_1
u_2
u_3
u_4
a

Theorem 2

Let the following conditions be satisfied:

- *transversality conditions*
- $\det K \neq 0$,
- $\exists \tilde{f} \in \mathbb{R} : |\mathbf{f}(t, \mathbf{x})| \leq \tilde{f}$, a.e. $t \in [a, b]$, all $\mathbf{x} \in \mathbb{R}^n$,
- $\gamma_i(\mathbf{x}+\mathbf{J}_i(t, \mathbf{x})) \leq \gamma_i(\mathbf{x})$ for all $(t, \mathbf{x}) \in [a, b] \times A$, $i = 1, \ldots, p$,
- $\exists \tilde{J}_i \in \mathbb{R}, i = 1, \ldots, p : |\mathbf{J}_i(t, \mathbf{x})| \leq \tilde{J}_i, (t, \mathbf{x}) \in [a, b] \times \mathbb{R}^n$,
- $\forall \varepsilon > 0 \; \exists \delta > 0 \; \forall \mathbf{x}, \mathbf{y} \in A$:
  $|\mathbf{x} - \mathbf{y}| < \delta \Rightarrow \|\mathbf{f}(\cdot, \mathbf{x}) - \mathbf{f}(\cdot, \mathbf{y})\|_\infty < \varepsilon$,
- $V \in \mathbb{C}([a_i, b_i]; \mathbb{R}^{n\times n}), i = 1, \ldots, p, V^* = \sup_{s\in[a,b]} |V(s)|$,
- $\mu_j \geq |K^{-1}| V^* \left(\tilde{f}(b - a) + \sum_{k=1}^{p} \tilde{J}_k\right)$
  corresponding
  $+2\tilde{f}(b - a) + \sum_{k=1}^{p} \tilde{J}_k + |K^{-1}\mathbf{c}|, \quad \rho_j \geq \tilde{f}, \quad j = 1, \ldots, n$.

Then the operator $\mathcal{G}$ is compact on $\overline{\Omega} \subset \mathbb{X}$ and has a fixed point in $\overline{\Omega} \subset \mathbb{X}$.

Theorem 3

Under the assumptions of Theorem 2 problem (1)–(3) *has at least one solution* **z** *such that*

$$\|\mathbf{z}\|_\infty \le \max\{\mu_1, \dots, \mu_n\}.$$

Dirichlet problem with one state-dependent impulse

We consider the second order Dirichlet boundary value problem with one state-dependent impulse

$$z''(t) = f(t, z(t)), \tag{9}$$

$$z(0) = 0, \quad z(T) = 0, \tag{10}$$

$$z'(\tau+) - z'(\tau-) = \mathcal{J}(z(\tau)), \quad \tau = \gamma(z(\tau)), \tag{11}$$

where we assume

$$f \in Car([0, T] \times \mathbb{R}), \quad \mathcal{J} \in C(\mathbb{R}), \quad \gamma \in C^1(\mathbb{R}), \tag{12}$$

$$\left\{ \begin{array}{l} \text{there exists } h \in Car([0, T] \times [0, \infty)) \text{ such that} \\ h(t, \cdot) \text{ is nondecreasing for a.e. } t \in [0, T] \text{ and} \\ |f(t, x)| \le h(t, |x|) \text{ for a.e. } t \in [0, T] \text{ and all } x \in \mathbb{R}, \end{array} \right. \tag{13}$$

$$\left\{ \begin{array}{l} \text{there exists } \mathcal{M} \in C([0, T]) \text{ nondecreasing} \\ \text{and such that } |\mathcal{J}(x)| \le \mathcal{M}(|x|) \text{ for } x \in \mathbb{R}. \end{array} \right. \tag{14}$$

Further, we assume

$$\exists K > 0 : \ \frac{1}{K}\left[\int_0^T h(s, K + T\mathcal{M}(K))\,\mathrm{d}s + \mathcal{M}(K)\right] < \min\left\{1, \frac{1}{T}\right\}. \tag{15}$$

$$\begin{cases} 0 < \gamma(x) < T, \quad |\gamma'(x)| < \dfrac{T}{K_1} \quad \text{for } |x| \le K_1, \\ \text{where } K_1 = K + T\mathcal{M}(K), \ K \text{ is from (15).} \end{cases} \tag{16}$$

Definition.

We say that $z : [0, T] \to \mathbb{R}$ is a solution of problem (9)–(11), if z is continuous on $[0, T]$, there exists unique $\tau \in (0, T)$ such that $\gamma(z(\tau)) = \tau$, $z|_{[0,\tau]}$ and $z|_{[\tau,T]}$ have absolutely continuous first derivatives, z satisfies equation (9) for a.e. $t \in [0, T]$ and fulfils conditions (10), (11).

Example

Consider problem (9)–(11) with
$T = 1, \quad f(t,x) = t^2 - |x|^\alpha \operatorname{sgn} x, \quad \mathcal{J}(x) = |x|^\beta \operatorname{sgn} x.$

- $\alpha, \beta \in (0,1) \quad \Longrightarrow \quad f$ and $\mathcal{J}$ are sublinear in x.
- Assumptions (13) and (14) are valid for

$$h(t,x) = t^2 + x^\alpha, \quad t \in [0,1], x > 0,$$

$$\mathcal{M}(x) = x^\beta, \quad x > 0.$$

- Assumption (15) is satisfied for any sufficiently large K.

Example

$$\lim_{x\to\infty} \frac{1}{x}\left[\int_0^1 h(s, x + \mathcal{M}(x))\,\mathrm{d}s + \mathcal{M}(x)\right]$$

$$= \lim_{x\to\infty} \frac{1}{x}\left[\frac{1}{3} + (x + x^\beta)^\alpha + x^\beta\right] = 0,$$

- $\alpha = \beta = \frac{1}{2} \implies K = 10$ and $K_1 = 10 + \sqrt{10}$.

Assumption (16): for $c \in (0, 1/(2K_1^2))$ we put

$$\gamma(x) = cx^2 + \frac{1}{2}, \quad x \in \mathbb{R}, \tag{17}$$

or for $c \in (0, 1/2)$, $n > cK_1$ we put

$$\gamma(x) = c \sin\frac{x}{n} + \frac{1}{2}, \quad x \in \mathbb{R}. \tag{18}$$

Example

Let us consider problem (9)–(11) with f and $\mathcal{J}$ having the linear behaviour in x and put

$$T = 1, \quad f(t,x) = a(t^\alpha - x), \quad \mathcal{J}(x) = bx, \quad a, b \in \mathbb{R}, \alpha > 0.$$

Then, assumptions (13) and (14) are valid for

$$h(t,x) = |a|(t^\alpha + x), \quad t \in [0,1], x > 0,$$

$$\mathcal{M}(x) = |b|x, \quad x > 0.$$

Example

$$\lim_{x\to\infty}\frac{1}{x}\left[\int_0^1 h(s,x+\mathcal{M}(x))\,\mathrm{d}s+\mathcal{M}(x)\right]$$
$$=\lim_{x\to\infty}\frac{1}{x}\left[|a|\left(\frac{1}{\alpha+1}+x(1+|b|)\right)+x|b|\right]$$
$$=|a|(1+|b|)+|b|.$$

Theorem 2 can be applied under the additional assumption

$$|a|<\frac{1-|b|}{1+|b|}. \tag{19}$$

If (19) holds, then for any sufficiently large K assumption (15) is satisfied. Then $K_1=K(1+|b|)$. Assumption (16) is fulfilled, if γ is given by (17) or (18).

Example

Let us consider problem (9)–(11) with f and $\mathcal{J}$ superlinear in x. Put

$$T = 1, \quad f(t,x) = c_1 t^3 + c_2 x^3, \quad \mathcal{J}(x) = \frac{1}{2}x^2, \quad c_1, c_2 \in \mathbb{R}. \qquad (20)$$

Then, assumptions (13) and (14) are valid for

$$h(t,x) = |c_1| t^3 + |c_2| x^3, \quad t \in [0,1], x > 0,$$

$$\mathcal{M}(x) = \frac{1}{2}x^2, \quad x > 0.$$

Example

$$\frac{1}{x}\left[\int_0^1 h(s,x+\mathcal{M}(x))\,\mathrm{d}s+\mathcal{M}(x)\right] = \frac{1}{x}\left[\frac{|c_1|}{4}+|c_2|\left(x+\frac{1}{2}x^2\right)^3+\frac{1}{2}x^2\right].$$

Assumption (15) is fulfilled provided there exists $K > 0$ such that

$$\frac{|c_1|}{4} + |c_2|\left(K + \frac{1}{2}K^2\right)^3 + \frac{1}{2}K^2 < K. \tag{21}$$

We search $K \in (0,1)$ fulfilling the equation

$$\left(\frac{27}{8}|c_2| + \frac{1}{2}\right)K^2 - K + \frac{|c_1|}{4} = 0.$$

Put $c_1 = 1$, $c_2 = -4/27$. Then we can choose $K = 1/2$ and $K_1 = 5/8$. Assumption (16) is fulfilled, if γ is given by (17) or (18).