Greatest and least solutions of measure differential equations

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(results from joint work with **Antonín Slavík**)

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Meanwhile, in my hometown...

...Carnival!!

Greatest and least solutions - History

G. Peano, *Sull' integrabilità delle equazioni differenziali di primo ordine*, Atti Acad. Sci. Torino 21 (1886), 677–685.

$$y'(t) = f(y(t), t), \quad y(a) = y_0$$

where $f(y, t)$ is real-valued and continuous on a neighborhood of (a, y_0)

The result:
the existence of solutions $y_{\min}, y_{\max} : [a, a + \delta] \to \mathbb{R}$ that are **extremal** in the sense that:

every other solution $y : [a, a + \delta] \to \mathbb{R}$ satisfies $y_{\min} \leq y \leq y_{\max}$

G. Monteiro and A. Slavík, *Extremal solutions of measure differential equations*, **submitted** (JMAA).

$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}g(s), \quad t \in [a, b] \tag{1}$$

where the integral is the Kurzweil-Stieltjes integral and g is a nondecreasing

G. Monteiro and A. Slavík, *Extremal solutions of measure differential equations*, **submitted** (JMAA).

$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}g(s), \quad t \in [a, b] \tag{1}$$

Definition

Let $z : I \subseteq [a, b] \to \mathbb{R}$ be a solution of Eq. (1).

We say that z is the greatest solution of Eq. (1) on I if any other solution $y : I \to \mathbb{R}$ satifies

$$y(t) \leq z(t) \quad \text{for every} \quad t \in I.$$

Symmetrically, we say that z is the least solution of Eq. (1) on I if any other solution $y : I \to \mathbb{R}$ satifies

$$z(t) \leq y(t) \quad \text{for every} \quad t \in I.$$

Existence of solution

M. Federson, J. G. Mesquita, A. Slavík, *Measure functional differential equations and functional dynamic equations on time scales*, J. Differential Equations 252 (2012), 3816–3847.

A. Slavík, *Well-posedness results for abstract generalized differential equations and measure functional differential equations*, J. Differential Equations 259 (2015), 666–707.

$$y(t) = y(t_0) + \int_{t_0}^{t} f(y_s, s)\,\mathrm{d}g(s), \quad t \in [t_0, t_0 + \sigma],$$

$$y_{t_0} = \phi.$$

Existence of solution

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\,\mathrm{d}g(s), \quad t \in [a, b]$$

$t_0 \in [a, b]$, $y_0 \in B \subseteq \mathbb{R}^n$
$g : [a, b] \to \mathbb{R}$ is BV and $f : B \times [a, b] \to \mathbb{R}^n$ satisfying:

(C1) For every $y \in B$, the integral $\int_a^b f(y, t)\,\mathrm{d}g(t)$ exists.

(C2) There exists $M : [a, b] \to \mathbb{R}$, which is K-S integrable w.r.t. g, such that

$$\left\| \int_u^v f(y, t)\,\mathrm{d}g(t) \right\| \leq \int_u^v M(t)\,\mathrm{d}g(t)$$

for every $y \in B$ and $[u, v] \subseteq [a, b]$.

(C3) For each $t \in [a, b]$, the mapping $y \in B \mapsto f(y, t)$ is continuous.

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\,\mathrm{d}g(s), \quad t \in [a, b]$$

$t_0 \in [a, b]$, $y_0 \in B \subseteq \mathbb{R}^n$
$g : [a, b] \to \mathbb{R}$ is BV and $f : B \times [a, b] \to \mathbb{R}^n$ satisfying:

(C1) For every $y : [a, b] \to B$ regulated, $\int_a^b f(y(t), t)\,\mathrm{d}g(t)$ exists.

(C2) There exists $M : [a, b] \to \mathbb{R}$, which is K-S integrable w.r.t. g, such that

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for every $y : [a, b] \to B$ regulated and $[u, v] \subseteq [a, b]$.

(C3) For each $t \in [a, b]$, the mapping $y \in B \mapsto f(y, t)$ is continuous.

Theorem

Assume

(i) $g : [a, b] \to \mathbb{R}$ *is BV*

(ii) $f : B \times [a, b] \to \mathbb{R}^n$ *satisfies conditions* (C1), (C2), (C3)

(iii) $y_+ = y_0 + f(y_0, t_0)\Delta^+ g(t_0)$ *and* $y_- = y_0 - f(y_0, t_0)\Delta^- g(t_0)$ *are interior points of* B

Then there exist $\delta_-, \delta_+ > 0$ *such that*

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\,\mathrm{d}g(s), \quad t \in [a, b]$$

has a solution on $[t_0 - \delta_-, t_0 + \delta_+] \cap [a, b]$

Lemma

Assume that $B \subset \mathbb{R}$ is open and

(i) $g : [a, b] \to \mathbb{R}$ *is BV*

(ii) $f : B \times [a, b] \to \mathbb{R}^n$ *satisfies conditions* (C1), (C2), (C3)

(iii) $y_+ = y_0 + f(y_0, t_0)\Delta^+ g(t_0)$ *and* $y_- = y_0 - f(y_0, t_0)\Delta^- g(t_0)$ *both belong to* B

Then:
there exist $\delta_-, \delta_+ > 0$ *such that each solution* $y : I \to B$ *of Eq.* (1), *I closed interval, can be extended to* $[t_0 - \delta_-, t_0 + \delta_+] \cap [a, b]$.

Lemma

Let $g : [a,b] \to \mathbb{R}$ *be BV and* $f : B \times [a,b] \to \mathbb{R}^n$*.*
Assume that $y : [t_0, T) \to \mathbb{R}^n$ *is a solution of*

$$y(t) = y_0 + \int_{t_0}^{t} f(y(s), s)\,\mathrm{d}g(s).$$

Then: y *can be extended to a solution on* $[t_0, T]$

if and only if

$y(T-)$ *exists and there is a vector* $\hat{y} \in B$ *such that*

$$y(T-) = \hat{y} - f(\hat{y}, T)\Delta^- g(T).$$

In this case, $y(T) = \hat{y}$*.*

Extremal solutions

$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}g(s), \quad t \in [a, b] \tag{1}$$

$f : B \times [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ is nondecreasing left continuous

$$y(t) = y_0 + \int_a^t f(y(s), s)\, \mathrm{d}g(s), \quad t \in [a, b] \qquad (1)$$

$f : B \times [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ is nondecreasing left continuous

(C4) If $u, v \in B$, with $u < v$, then

$$u + f(u, t)\Delta^+ g(t) \le v + f(v, t)\Delta^+ g(t) \text{ for every } t \in [a, b].$$

In this case:

if $y_1, y_2 : I \to B$ are solutions of Eq.(1) with $y_1(\tau) \le y_2(\tau)$

$$y_1(\tau+) = y_1(\tau) + f(y_1(\tau), \tau)\Delta^+ g(\tau) \le y_2(\tau) + f(y_2(\tau), \tau)\Delta^+ g(\tau) = y_2(\tau+)$$

$$y(t) = y_0 + \int_a^t f(y(s), s)\, \mathrm{d}g(s), \quad t \in [a, b] \qquad (1)$$

$f : B \times [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}$ is nondecreasing left continuous

(C4) If $u, v \in B$, with $u < v$, then

$$u + f(u, t)\Delta^+ g(t) \leq v + f(v, t)\Delta^+ g(t) \text{ for every } t \in [a, b).$$

Theorem

Assume that $B \subset \mathbb{R}$ is closed and f satisfies conditions (C1), (C2), (C3), (C4).
If Eq. (1) has a solution on $[a, b]$, then it has the greatest/least solution on $[a, b]$.

Peano's uniqueness theorem

(C2′) For each compact set $C \subseteq B$, there exists $M : [a,b] \to \mathbb{R}$, which is K-S integrable w.r.t. g, such that

$$\left| \int_u^v f(y,t)\,\mathrm{d}g(t) \right| \leq \int_u^v M(t)\,\mathrm{d}g(t)$$

for every $y \in C$ and $[u,v] \subseteq [a,b]$.

Theorem

Assume that $B \subset \mathbb{R}$ is closed and f satisfies conditions (C1), (C2′), (C3), (C4).
If the function f is nonincreasing in the first variable, then

$$y(t) = y_0 + \int_a^t f(y(s),s)\,\mathrm{d}g(s), \quad t \in [a,b]$$

has at most one solution on $[a,b]$.

Theorem

Assume

(i) f *satisfies conditions* (C1), (C2′), (C3), (C4)

(ii) $y_+ = y_0 + f(y_0, t_0)\Delta^+ g(t_0)$ *is an interior points of* B

Then there exists $\delta > 0$ *such that*

1. *Eq.* (1) *has the greatest solution* $y_{\max}$ *and the least solution* $y_{\min}$ *in* $B \times [a, a+\delta]$.
2. *For any solution* $y : I \to B$ *of Eq.* (1), $a \in I \subsetneq [a, a+\delta]$, *we have*

$$y_{\min}(t) \leq y(t) \leq y_{\max}(t) \quad \text{for all } t \in I.$$

Definition

Let $I \subseteq [a, b]$ be an interval with $a \in I$.
A regulated function $\alpha : I \to B$ is a lower solution of Eq. (1) on I if $\alpha(a) \leq y_0$ and

$$\alpha(v) - \alpha(u) \leq \int_u^v f(\alpha(s), s)\,\mathrm{d}g(s), \quad [u, v] \subseteq I. \tag{1}$$

Symmetrically, a regulated function $\beta : I \to B$ is an upper solution of Eq. (1) on I if $\beta(a) \geq y_0$ and

$$\beta(v) - \beta(u) \geq \int_u^v f(\beta(s), s)\,\mathrm{d}g(s), \quad [u, v] \subseteq I. \tag{2}$$

Theorem

Let $B \subset \mathbb{R}$ be open and assume that

(i) *f satisfies conditions* (C1), (C2′), (C3), (C4)

(ii) $y_+ = y_0 + f(y_0, t_0)\Delta^+ g(t_0) \in B$

If $y_{\max} : I \to B$ and $y_{\min} : J \to B$, where $a \in I \subseteq [a, b]$ and $a \in J \subseteq [a, b]$, are the noncontinuable extremal solutions of Eq. (1), *then*

1. *If $\alpha : I' \to B$, where $a \in I' \subseteq I$, is a lower solution, then*
$$\alpha(t) \leq y_{\max}(t) \quad \text{for all } t \in I'.$$

2. *If $\beta : J' \to B$, where $a \in J' \subseteq J$, is an upper solution, then*
$$\beta(t) \geq y_{\min}(t) \quad \text{for all } t \in J'.$$

Consequently,

$$y_{\max}(t) = \max\{\alpha(t);\ \alpha \text{ is a lower solution on } [a, t]\}, \quad t \in I,$$
$$y_{\min}(t) = \min\{\beta(t);\ \beta \text{ is an upper solution on } [a, t]\}, \quad t \in J.$$

references

Application to impulsive systems

$$
\begin{aligned}
y'(t) &= f(y(t), t), \quad \text{a.e. in } [a, b], \\
\Delta^+ y(t_k) &= I_k(y(t_k)), \quad k \in \{1, \dots, m\}, \\
y(a) &= y_0,
\end{aligned}
$$

where $a \leq t_1 < \cdots < t_m < b$, $f : B \times [a, b] \to \mathbb{R}^n$, $I_1, \dots, I_m : B \to \mathbb{R}^n$

$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}s + \sum_{k;\ t_k<t} I_k(y(t_k)), \quad t \in [a, b],$$

where $a \leq t_1 < \cdots < t_m < b$, $f : B \times [a, b] \to \mathbb{R}^n$, $I_1, \ldots, I_m : B \to \mathbb{R}^n$

Application to impulsive systems

$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}s + \sum_{k;\ t_k<t} I_k(y(t_k)), \quad t \in [a, b],$$

where $a \leq t_1 < \cdots < t_m < b$, $f : B \times [a, b] \to \mathbb{R}^n$, $I_1, \dots, I_m : B \to \mathbb{R}^n$

Define:

$$g(s) = s + \sum_{k=1}^{m} \chi_{(t_k,\infty)}(s), \quad s \in [a, b]$$

$$\tilde{f}(z, t) = \begin{cases} f(z, t), & t \in [a, b] \backslash \{t_1, \dots, t_m\}, \\ I_k(z), & t = t_k \ \text{ for some } \ k \in \{1, \dots, m\}. \end{cases}$$

$$z(t) = y_0 + \int_a^t \tilde{f}(z(s), s)\,\mathrm{d}g(s), \quad t \in [a, b].$$

references

$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}s + \sum_{k;\ t_k<t} I_k(y(t_k)), \quad t \in [a, b],$$

(C1) For every $y \in B$, the integral $\int_a^b \tilde{f}(y, t)\,\mathrm{d}g(t)$ exists.

$$z(t) = y_0 + \int_a^t \tilde{f}(z(s), s)\,\mathrm{d}g(s), \quad t \in [a, b].$$

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$$\left\| \int_u^v \tilde{f}(y, t)\,\mathrm{d}g(t) \right\| \le \int_u^v M(t)\,\mathrm{d}g(t)$$

for every $y \in B$ and $[u, v] \subseteq [a, b]$;

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$$y(t) = y_0 + \int_a^t f(y(s),s)\,\mathrm{d}s + \sum_{k;\ t_k<t} I_k(y(t_k)), \quad t \in [a,b],$$

(C1) For every $y \in B$, the integral $\int_a^b f(y,t)\,\mathrm{d}t$ exists.

(C2) There exists $M : [a,b] \to \mathbb{R}$ K-H integrable and $m_k > 0$ such that

$$\left\| \int_u^v f(y,t)\,\mathrm{d}t \right\| \leq \int_u^v M(t)\,\mathrm{d}t$$

for every $y \in B$ and $[u,v] \subseteq [a,b]$; $\|I_k(y)\| \leq m_k$ for $y \in B$.

$$z(t) = y_0 + \int_a^t \tilde{f}(z(s),s)\,\mathrm{d}g(s), \quad t \in [a,b].$$

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for every $y \in B$ and $[u,v] \subseteq [a,b]$; $\|I_k(y)\| \le m_k$ for $y \in B$.

(C3) For each $t \in [a,b]$, the mapping $y \in B \mapsto \tilde{f}(y,t)$ is continuous.

$$z(t) = y_0 + \int_a^t \tilde{f}(z(s), s)\,\mathrm{d}g(s), \quad t \in [a, b].$$

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for every $y \in B$ and $[u, v] \subseteq [a, b]$; $\|I_k(y)\| \leq m_k$ for $y \in B$.

(C3) For each $t \in [a, b] \backslash \{t_1, \dots, t_m\}$, $y \mapsto f(y, t)$ is continuous in B; and $I_k : B \to \mathbb{R}^n$ is continuous for each $k \in \{1, \dots, m\}$.

$$z(t) = y_0 + \int_a^t \tilde{f}(z(s), s)\,\mathrm{d}g(s), \quad t \in [a, b].$$

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$$y(t) = y_0 + \int_a^t f(y(s), s)\,\mathrm{d}s + \sum_{k;\ t_k < t} I_k(y(t_k)), \quad t \in [a, b],$$

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Thank you!

Děkuju!