On Oscillatory Solutions of Higher Order Nonlinear Nonautonomous Differential Equations and Systems

Ivan Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: `kig@rmi.ge`

On an infinite interval $[a,+\infty[$, we consider the differential system

$$u_1^{(n_1)} = f_1(t,u_2), \quad u_2^{(n_2)} = f_2(t,u_1) \tag{1}$$

and the differential equation

$$u^{(n)} = f(t,u), \tag{2}$$

where $n_1 \geq 1$, $n_2 \geq 2$, $n > 2$, $a > 0$, while $f_i : [a,+\infty[\,\times\mathbb{R} \to \mathbb{R}$ $(i=1,2)$ and $f : [a,+\infty[\,\times\mathbb{R} \to \mathbb{R}$ are continuous functions.

We have investigated oscillation properties of solutions of system (1) in the case where one of the following two conditions

$$f_i(t,0) = 0, \quad (-1)^{i-1} f_i(t,x) \leq (-1)^{i-1} f_i(t,y) \;\text{ for }\; t \geq a, \;\; x < y \;\; (i=1,2) \tag{3}$$

or

$$f_i(t,0) = 0, \;\; f_i(t,x) \leq f_i(t,y) \;\text{ for }\; t \geq a, \;\; x < y \;\; (i=1,2) \tag{4}$$

is satisfied. As for equation (2), it has been studied for the case where the function f satisfies either the condition

$$f(t,0) = 0, \;\; f(t,x) \geq f(t,y) \;\text{ for }\; t > a, \;\; x < y \tag{5}$$

or the condition

$$f(t,0) = 0, \;\; f(t,x) \leq f(t,y) \;\text{ for }\; t > a, \;\; x < y. \tag{6}$$

A solution of system (1) (of equation (2)) defined on some interval $[a_0,+\infty[\,\subset[a,+\infty[$ is said to be **proper** if it does not identically equal to zero in any neighbourhood of $+\infty$.

A proper solution (u_1,u_2) of system (1) is said to be **oscillatory** if at least one of its components changes sign in any neighbourhood of $+\infty$, and is said to be **Kneser** one if in the interval $[a_0,+\infty[$ it satisfies the inequalities

$$\begin{gathered}(-1)^i u_1^{(i)}(t) u_1(t) \geq 0 \quad (i=1,\dots,n_1),\\ (-1)^k u_2^{(k)}(t) u_2(t) \geq 0 \;\; (k=1,\dots,n_2).\end{gathered}$$

A proper solution u of equation (2) is said to be **oscillatory** if it changes sign in any neighbourhood of $+\infty$ and is said to be **Kneser** one if in the interval $[a_0,+\infty[$ satisfies the inequalities

$$(-1)^i u^{(i)}(t) u(t) \geq 0 \quad (i=1,\dots,n).$$

Assume

$$n_0 = n_1 + n_2$$

and introduce

Definition 1. System (1) has the property A_0 if every its proper solution for even n_0 is oscillatory, and for odd n_0 either is oscillatory or Kneser one.

Definition 2. System (2) has the property B_0 if every its proper solution for even n_0 is either oscillatory, or Kneser one, or satisfies the condition

$$\lim_{t\to+\infty} |u_1^{(n_1-1)}(t)| = +\infty, \quad \lim_{t\to+\infty} |u_2^{(n_2-2)}(t)| = +\infty.$$

If m is a natural number, then by $\mathcal{N}_m^0$ we denote the set of those $k \in \{1,\dots,m\}$ for which $m+k$ is even.

For an arbitrary natural k, we put

$$I_k(t,x) = x\left[t^{n_1-1} + \int_a^t (t-s)^{n_1-1}\big|f_1(s,xs^{k-1})\big|\,ds\right].$$

Theorem 1. *Let condition* (3) *be satisfied and for any $x \neq 0$ and $k \in \mathcal{N}_{n_2-1}^0$ the equalities*

$$\int_a^{+\infty} |f_1(t,x)|\,dt = \int_a^{+\infty} t^{n_2-1}|f_2(t,x)|\,dt = +\infty, \tag{7}$$

$$\int_a^{+\infty} t^{n_2-k-1}\big|f_2(t,I_k(t,x))\big|\,dt = +\infty \tag{8}$$

be fulfilled. Then system (1) *has the property A_0.*

Theorem 2. *Let condition* (4) *be satisfied. If, moreover, $n_2 > 2$ ($n_2 = 2$) and for any $x \neq 0$ and $k \in \mathcal{N}_{n_2-2}^0$ equalities* (7) *and* (8) *hold (for any $x \neq 0$ equality* (7) *is fulfilled), then system* (1) *has the property B_0.*

Remark. Let the conditions of Theorem 1 be satisfied and on the set $[a,+\infty[\,\times\mathbb{R}$ the inequality

$$\sum_{i=1}^{2} |f_i(t,x)| \le h(t)|x| \tag{9}$$

be fulfilled, where $h : [a,+\infty[\,\to [0,+\infty[$ is a continuous function. Then it is evident that for the even n_0, every nontrivial solution of system (1) is proper and oscillatory, and for the odd n_0, this system has an infinite set of proper oscillatory and Kneser solutions.

If along with the conditions of Theorem 2, condition (9) is likewise fulfilled, then the question on the existence of proper oscillatory solutions of system (1) requires an additional investigation. In this connection, there arises the problem on the existence of a solution of system (1) satisfying the boundary conditions

$$\begin{gathered} u_1^{(i-1)}(a) = c_i u_2^{(n_2-1)}(a) + c_{0i} \ \ (i=1,\dots,n_1), \quad u_2^{(k-1)}(a) = \\ = c_{n_1+k} u_2^{(n_2-1)}(a) + c_{0n_1+k} \ \ (k=1,\dots,n_2-1), \quad \limsup_{t\to+\infty} |u_2^{(n_2-2)}(t)| < +\infty, \end{gathered} \tag{10}$$

where c_i and c_{0i} $(i=1,\dots,n_0-1)$ are real constants, and

$$c_i > 0 \ \ (i=1,\dots,n_0-1). \tag{11}$$

Lemma 1. *Let conditions* (4), (9), (11) *and*

$$\int\limits_a^{+\infty} \left|f_1(t, xt^{n_2-1})\right| dt = +\infty \quad \textit{for} \ x \neq 0 \tag{12}$$

are fulfilled. Then problem (1), (10) *has at least one solution. If, however, along with* (4), (11) *and* (12), *the condition*

$$\sum_{i=1}^{2} \left|f_i(t,x) - f_i(t,y)\right| \leq h(t)|x-y| \tag{13}$$

is fulfilled, then problem (1), (10) *has one and only one solution.*

From Theorem 2 and Lemma 1 it follows

Theorem 3. *Let the conditions of Theorem* 2 *hold, and*

$$c_i > 0 \ \ (i = 1, \dots, n_0 - 1), \ \ c_{01} = 0, \ \ \sum_{i=2}^{n_0-1} |c_{0i}| > 0.$$

If, moreover, condition (9) (*condition* (13)) *is fulfilled, then problem* (1), (10) *has at least one* (*one and only one*) *solution and it is oscillatory.*

For equation (2), Definitions 1, 2 and Theorems 1–3 have the following forms.

Definition 3. Equation (2) has property A_0 if any proper solution of this equation in case n even is oscillatory and in case n odd either is oscillatory or is a Kneser solution.

Definition 4. Equation (2) has property B_0 if any proper solution of this equation in case n even either is oscillatory, or is a Kneser solution, or satisfies the condition

$$\lim_{t\to+\infty} |u^{(n-2)}(t)| = +\infty, \tag{14}$$

and in case n odd either is oscillatory or satisfies condition (14).

Theorem 4. *If along with* (5) *the condition*

$$\int\limits_a^{+\infty} t^{n-k-1}\left|f(t, xt^{k-1})\right| dt = +\infty \quad \textit{for} \ x \neq 0, \ \ k \in \mathcal{N}_{n-1}^0 \tag{15}$$

holds, then equation (2) *has property* A_0.

Theorem 5. *If* $n \geq 3$ *and along with* (6) *the condition*

$$\int\limits_a^{+\infty} t^{n-k-1}\left|f(t, xt^{k-1})\right| dt = +\infty \quad \textit{for} \ x \neq 0, \ \ k \in \mathcal{N}_{n-2}^0 \tag{16}$$

holds, then equation (2) *has property* B_0.

Theorem 6. *Let the conditions of Theorem* 5 *hold and* c_i *and* c_{0i} $(i = 1, \dots, n)$ *are the real constants such that*

$$c_i > 0 \ \ (i = 1, \dots, n - 1), \ \ c_{01} = 0, \ \ \sum_{i=2}^{n-1} |c_{0i}| > 0. \tag{17}$$

If, moreover, on the set $[a,+\infty[$ *the condition*

$$|f(t,x)| \le h(t)|x| \quad \big(|f(t,x)-f(t,y)| \le h(t)|x-y|\big)$$

is fulfilled, where $h : [a,+\infty[\to [0,+\infty[$ *is a continuous function, then equation* (2) *has at leat one* (*one and only one*) *oscillatory solution such that*

$$u^{(i-1)}(a) = c_i u^{(n-1)}(a) + c_{0i} \;\; (i=1,\dots,n-1). \tag{18}$$

The conditions of Theorems 4 and 5 are in a certain sense unimprovable. Moreover, the following statements are valid.

Theorem 7. *Let condition* (5) *be satisfied and for any* $x \neq 0$ *there exist numbers* $t_x \ge a$ *and* $\delta(x) > 0$ *such that*

$$t^{n-k-1}\big|f(t,xt^{k-1})\big| \ge \delta(x)\big|f(t,xt^{n-1})\big| \;\; \text{for} \;\; t \ge t_x, \;\; k \in \mathcal{N}_{n-1}^0.$$

Then for the differential equation (2) *to have property* A_0 *it is necessary and sufficient equality* (15) *to be fulfilled.*

Theorem 8. *Let conditions* (6) *be fulfilled,* $n \ge 3$ *and for any* $x \neq 0$ *there exist numbers* $t_x \ge a$ *and* $\delta(x) > 0$ *such that*

$$t^{n-k-2}\big|f(t,xt^{k-1})\big| \ge \delta(x)\big|f(t,xt^{n-2})\big| \;\; \text{for} \;\; t \ge t_x, \;\; k \in \mathcal{N}_{n-2}^0.$$

Then for the differential equation (2) *to have property* B_0 *it is necessary and sufficient equality* (16) *to be fulfilled.*

An essential difference between the above formulated theorems and the results obtained earlier (see, e.g., [1]–[3] and the references therein) is that they cover the case, where the right-hand sides of system (1) and of equation (2) are slowly increasing with respect to the phase variable functions.

As an example, let us consider the differential equation

$$u^{(n)} = g_0(t)f_0(u) + g_1(t)\ln\big(1+|u|\big)\operatorname{sign}(u), \tag{19}$$

$g_i : [a,+\infty[\to \mathbb{R}$ $(i=0,1)$ are continuous functions, $f_0 : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function such that

$$f_0(x)x > 0 \;\text{ for }\; x \neq 0, \quad \sup\big\{|f_0(x)| : \; x \in \mathbb{R}\big\} < +\infty.$$

Theorems 6–8 result in the following corollaries.

Corollary 1. *If* $n \ge 3$ *and* $g_0(t) \le 0$, $g_1(t) \le 0$ *for* $t \ge a$, *then for equation* (19) *to have property* A_0 *it is necessary and sufficient the equality*

$$\int_a^{+\infty} \big[g_0(t) + g_1(t)\ln t\big]\,dt = -\infty$$

to be fulfilled.

Corollary 2. *If* $n \ge 4$ *and* $g_0(t) \ge 0$, $g_1(t) \ge 0$ *for* $t \ge a$, *then for differential equation* (19) *to have property* B_0 *it is necessary and sufficient the equality*

$$\int_a^{+\infty} t\big[g_0(t) + g_1(t)\ln t\big]\,dt = +\infty \tag{20}$$

to be satisfied. Moreover, if along with (20) *the condition* (17) *is fulfilled* (*and the function* f_0 *satisfies the Lipschitz condition*), *then the problem* (19), (18) *has at least one* (*one and only one*) *oscillatory solution.*

Acknowledgement

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

References

[1] I. Kiguradze, On the oscillation of solutions of the equation $d^m u/dt^m + a(t)|u|^n \operatorname{sign} u = 0$. (Russian) *Mat. Sb.* **65** (1964), No. 2, 172–187.

[2] I. Kiguradze, Concerning the oscillation of solutions of nonlinear differential equations. (Russian) *Differentsial'nye Uravneniya* **1** (1965), No. 8, 995–1006; English transl.: *Differ. Equations* **1** (1965), 773–782.

[3] I. Kiguradze and T. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. *Springer Science* & *Business Media*, 2012.