Hardy Inequality, Compact Embeddings and Properties of Certain Eigenvalue Problems

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Radial solutions :

$$(1) \quad \begin{cases} -\operatorname{div}\left(a(|x|)\nabla u\right) = \lambda\, b(|x|)\, u & \text{in } B_R(0) \\ u = 0 & \text{on } \partial B_R(0) \end{cases}$$

eigenvalues, eigenfunctions

$a(|x|)$, $b(|x|)$ degenerate / blow up near the bdry

$a(|x|) = (R-|x|)^{\alpha} \qquad b(|x|) = (R-|x|)^{\beta}$

$\alpha, \beta > 0$ degenerate $\qquad \alpha, \beta < 0$ blow up

spherical coordinates :

$$(2) \quad \begin{cases} -\left(r^{N-1}(R-r)^{\alpha} u'\right)' = \lambda\, (R-r)^{\beta} u & \text{in } (0,R) \\ u'(0) = u(R) = 0 \end{cases}$$

(2) is a special case of:

$$(3)\begin{cases} \left(\rho(t)\,|u'(t)|^{p-2}u'(t)\right)' + \lambda\,\sigma(t)\,|u(t)|^{p-2}u(t) = 0, \quad t \in (a,b) \\ \lim\limits_{t\to a_+} \rho(t)\,|u'(t)|^{p-2}u'(t) = \lim\limits_{t \to b_-} u(t) = 0 \end{cases}$$

$p > 1$, $-\infty \leqq a < b \leqq +\infty$ $\quad \rho, \sigma$ continuous, positive

$$\forall x \in (a,b): \ \sigma \in L^1(a,x), \ \rho^{1-p'} \in L^1(x,b)$$

NOTE: $\sigma, \rho^{1-p'} \notin L^1(a,b)$ in general!

Weighted spaces:

$L^p(\sigma)$: $\quad \|u\|_{p;\sigma} = \left(\int_a^b \sigma(t)\,|u(t)|^p\,dt\right)^{\frac{1}{p}} < \infty$

$W_b^{1,p}(\rho)$: $\quad u$ absolutely cont. on every compact subint. (a,b)

$u(b) = 0$ and

$$\|u\|_{1,p;\rho} = \left(\int_a^b \rho(t)\,|u'(t)|^p\,dt\right)^{\frac{1}{p}} < \infty$$

$$W_b^{1,p}(\rho) \hookrightarrow L^p(\sigma) \iff \sup_{t\in(a,b)} \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} < \infty$$

$$W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma) \iff \underbrace{\lim_{\substack{t\to a_+\\ t\to b_-}} \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} = 0}_{(4)}$$

<u>THEOREM</u> (P.D. + K. Kuliev): (i) $\iff$ (ii) $\iff$ (iii)

(i) (3) has "Sturm-Liouville" property

(ii) $W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma)$

(iii) (4) hold true

$u = u(t)$ is nonoscillatory solution of eq. (3):

$\exists\, \bar{a}, \bar{b} \in (a,b): \; u(t) \neq 0 \quad \forall t \in (a,\bar{a}) \cup (\bar{b}, b)$

u is called oscillatory othewise

Eq. (3) is called nonoscillatory if any of its solutions is nonoscillatory.

Let

$$\limsup_{\substack{t \to a_+ \\ t \to b_-}} \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} < \frac{1}{\lambda} \cdot \frac{(p-1)^{p-1}}{p^p}$$

$$\Downarrow$$

Eq. (3) is nonoscillatory

Let

$$\limsup_{\substack{t \to a_+ \\ t \to b_-}} \left(\int_a^t r(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} > \frac{1}{\lambda}$$

$$\Downarrow$$

any solution of (3) is osaillatory

[cf. O. Došlý: Czech. Math. J. 50 (2000), 657-671]

$$\lim_{\substack{t \to a_+ \\ t \to b_-}} \left(\int_a^t \sigma(\tau)\, d\tau \right) \cdot \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau \right)^{p-1} = 0 \quad \Rightarrow$$

$\Rightarrow$ $\forall \lambda > 0$ eq. (3) is nonoscillatory

$$\lim_{t \to a_+} \left(\int_a^t \sigma(\tau)\, d\tau \right) \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau \right)^{p-1} \neq 0$$

or $\Rightarrow$

$$\lim_{t \to b_-} \quad \text{—— " ——} \quad \neq 0$$

$\Rightarrow$ $\exists \lambda_0 > 0$ $\forall \lambda \geq \lambda_0$ any solution of eq. (3) is oscillatory

$$W_b^{1,p}(\rho) \hookrightarrow L^p(\sigma) \iff \sup_{t\in(a,b)} \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} < \infty$$

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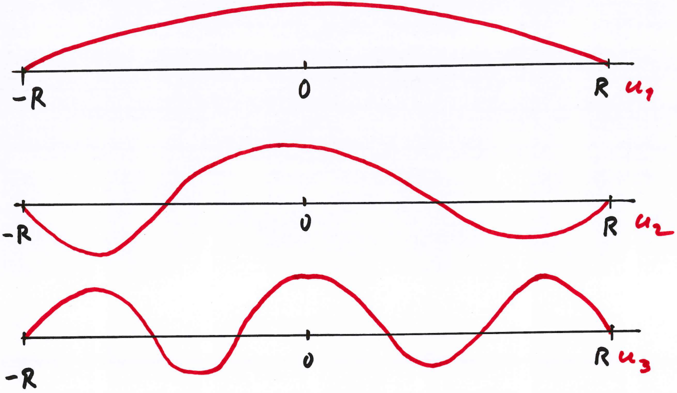
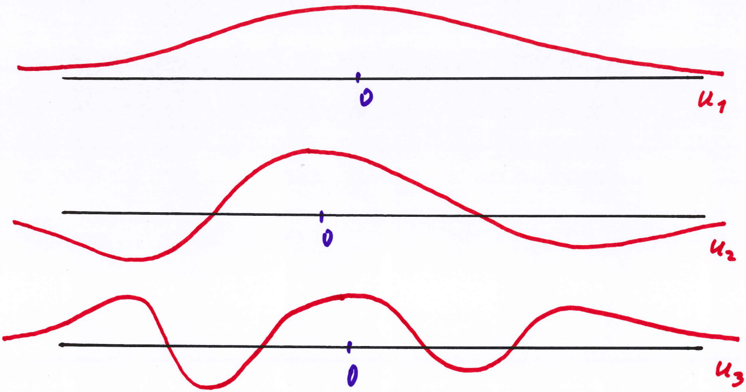
Profile of radial eigenfunctions: $R < +\infty$

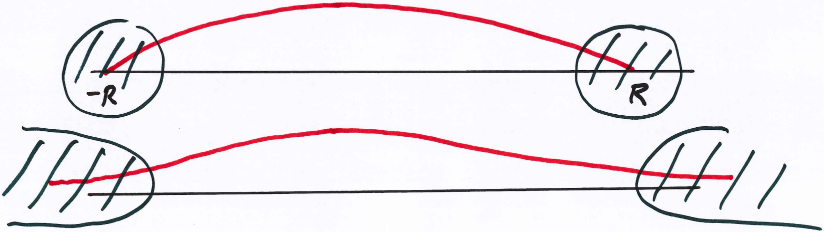


etc.

Profile of eigenfunctions : $R = +\infty$



etc.



We concentrate on the rate of the decay of eigenfunctions near $R < +\infty$ and near $+\infty$.

"How the degeneration/blow up of the coefficient corresponds to the Hopf-type maximum principle at the boundary $\partial B_R(0)$?"

P. D. + A. Kufner + K. Kuliev: Proc. Steklov Inst. Math. 2014 (Vol. 284), 148-154

Let $\exists\, \varepsilon \in (0, p-1)$, $c > 0$: $\forall t \in (a,b)$

$$(5) \qquad \left(\int_a^t \sigma(\tau)\, d\tau\right) \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau\right)^{p-1} \le c \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau\right)^{\varepsilon}.$$

Then $\exists\, c_1, c_2 > 0$ $\exists\, \bar{b} \in (a,b)$: $\forall t \in (\bar{b}, b)$:

$$(6) \qquad c_1 \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau\right) \le |u(t)| \le c_2 \left(\int_t^b \rho^{1-p'}(\tau)\, d\tau\right).$$

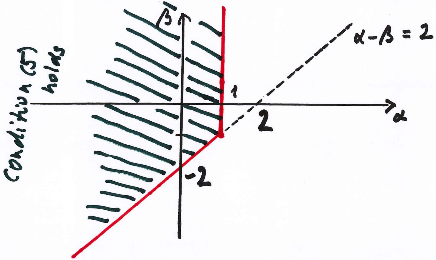
In other words: „The asymptotics of $u(t)$ near b is given by $\int_t^b \rho^{1-p'}(\tau)\, d\tau$."

Consider the problem

$$\begin{cases} -\operatorname{div}\left((R-|x|)^{\alpha}\nabla u\right) = \lambda (R-|x|)^{\beta} u & \text{in } B_R(0) \\ u = 0 & \text{on } \partial B_R(0) \end{cases}$$

Condition (5) is satisfied if

$\beta < -1$ and $\alpha - \beta < 2$ or $\beta \geq -1$ and $\alpha < 1$



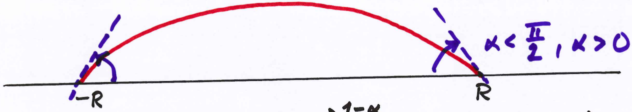
$$(6) \Rightarrow \quad \exists\, \bar{R} \in (0,R) \quad \forall x \in B_R(0) : |x| \in (\bar{R}, R)$$

$$\exists\, c_1, c_2 > 0$$

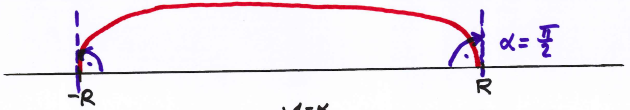
$$c_1 (R-|x|)^{1-\alpha} \le |u(x)| \le c_2 (R-|x|)^{1-\alpha}$$

This estimate generalizes (for $\alpha \neq 0$) and recovers (for $\alpha = 0$) Hopf maximum principle at $\partial B_R(0)$.

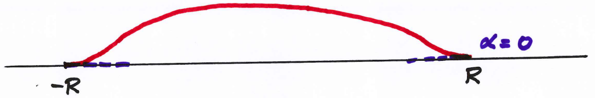
$\alpha = 0 \implies u(x) \approx (R - |x|)$ as $x \to \partial B_R(0)$:



$\alpha \in (0,1) \implies u(x) \approx (R - |x|)^{1-\alpha}$ as $x \to \partial B_R(0)$



$\alpha < 0 \implies u(x) \approx (R - |x|)^{1-\alpha}$ as $x \to \partial B_R(0)$



$\alpha = 0 \quad \Rightarrow \quad \dfrac{\partial u}{\partial \nu} < 0$ Hopf m.p.

$\alpha \in (0,1)$, i.e. coefficient degenerates towards the boundary

$$\Rightarrow \quad \frac{\partial u}{\partial \nu} = -\infty$$

$\alpha < 0$, i.e. coefficient is singular (blows up) towards the boundary

$$\Rightarrow \quad \frac{\partial u}{\partial \nu} = 0$$

P. D. + A. Kufner + K. Kuliev: Proc. Steklov Inst. Math. 2014 (Vol. 284), 148-154

Let $\exists\, \varepsilon \in (0, p-1),\ c > 0 \ :\ \forall t \in (a,b)$

$$(5) \qquad \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} \le c\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{\varepsilon}.$$

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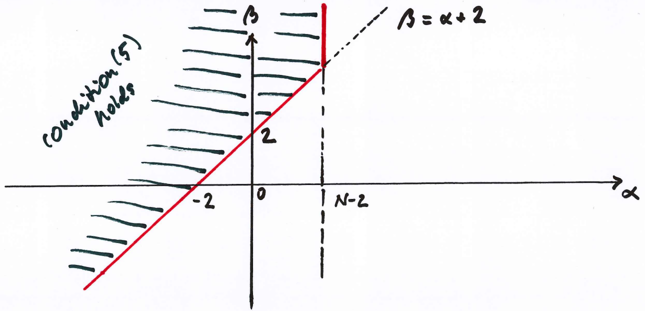
Consider the problem

$$\begin{cases} -\operatorname{div}\left(\dfrac{1}{(1+|x|)^{\alpha}}\nabla u\right) = \lambda \dfrac{1}{(1+|x|)^{\beta}} u \quad \text{in } \mathbb{R}^N \\ \lim\limits_{|x|\to\infty} u(x) = 0 \end{cases}$$

Condition (5) is satisfied if $\alpha + 2 < \min\{N, \beta\}$



$$(6) \Rightarrow \exists \bar{R} > 0 \ \exists c_1, c_2 > 0 \ \forall x \in \mathbb{R}^N \setminus B_{\bar{R}}(0)$$

$$\frac{c_1}{|x|^{N-(\alpha+2)}} \le |u(x)| \le \frac{c_2}{|x|^{N-(\alpha+2)}}$$

In particular, if

$\alpha = 0$ and $\beta > 2$: $\quad u(x) \approx \dfrac{c}{|x|^{N-2}}, \ |x| \to \infty$

$\alpha < -2$ and $\beta = 0$: $\quad u(x) \approx \dfrac{c}{|x|^{N-(\alpha+2)}}, \ |x| \to \infty$

$$W_b^{1,p}(\rho) \hookrightarrow L^p(\sigma) \iff \sup_{t\in(a,b)} \left(\int_a^t \sigma(\tau)\,d\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,d\tau\right)^{p-1} < \infty$$

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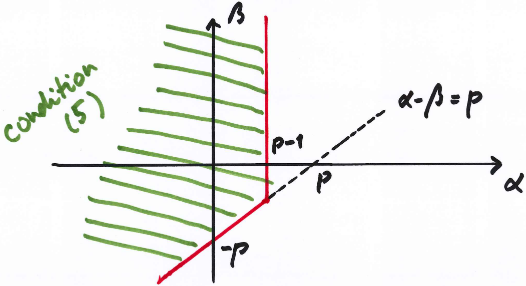

$(6) \Rightarrow \quad \exists\, \bar{R} \in (0, R) \;\; \forall x \in B_R(0) \setminus B_{\bar{R}}(0)$

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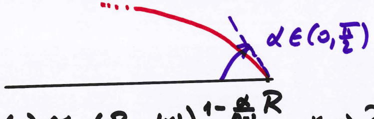
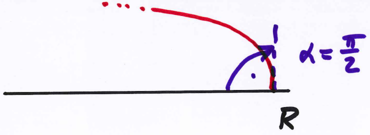
This estimate generalizes (for $\alpha \neq 0$) and recovers (for $\alpha = 0$) Vázquez maximum principle at $\partial B_R(0)$.

$\alpha = 0 \Rightarrow u(x) \simeq (R - |x|), \quad x \to \partial B_R(0)$



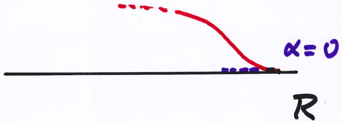
$\frac{\partial u}{\partial \nu} < 0$

$\alpha \in (0, p-1) \Rightarrow u(x) \simeq (R - |x|)^{1 - \frac{\alpha}{p-1}}, \quad x \to \partial B_R(0)$



$\frac{\partial u}{\partial \nu} = -\infty$

$\alpha < 0 \Rightarrow \dots$



$\frac{\partial u}{\partial \nu} = 0$

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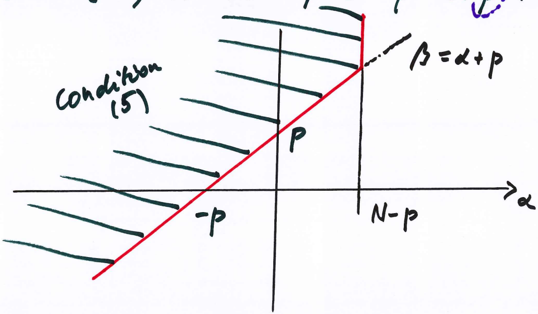

$$(6) \Rightarrow \exists \bar{R}>0 \ \exists c_1, c_2 > 0 \ \forall x \in \mathbb{R}^N \setminus B_{\bar{R}}(0)$$

$$\frac{c_1}{|x|^{\frac{N-(\alpha+p)}{p-1}}} \le |u(x)| \le \frac{c_2}{|x|^{\frac{N-(\alpha+p)}{p-1}}}$$

In particular, if

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$\alpha < -p$ and $\beta = 0$: $\quad u(x) \approx \dfrac{c}{|x|^{\frac{N-(\alpha+p)}{p-1}}}$


KČT
ŠUKAČKA
887 m
2010
KT192m
PĚŠÍ TRASA KČT
SUCHÉ STUDÁNKY (ROZC.) 1 km
DĚPOLTICE (ROZC.) 6 km
DĚPOLTICE 6,5 km
2007
KT192a
LESY ČR
ZELENÁ
PRO VAŠE KROKY
podporujeme značení turistických tras
Horská služba
Mountain Rescue Service
Bergrettungsdienst
NON STOP

Thank you!