p-Trigonometric and p-Hyperbolic Functions in Real and Complex Domain

Lukáš Kotrla

For $p > 1$, p-trigonometric and p-hyperbolic functions are generalization of well-known trigonometric and hyperbolic functions. Although many properties (e.g. continuity, oddness/evenness, periodicity, reflexivity) of p-trigonometric functions are similar to the properties of classical trigonometric functions, there is fundamental difference in differentiability. Indeed, function $\sin_p$ is only in space $C^1(\mathbb{R})$. Detailed study of the order of differentiability reveals that $\sin_p \in C^\infty\left(\frac{-\pi_p}{2}, \frac{\pi_p}{2}\right)$ for p even and, in this special case, $\sin_p$ can be expressed as Maclaurin series, which converges on $\left(\frac{-\pi_p}{2}, \frac{\pi_p}{2}\right)$. It allows us naturally extend $\sin_p$ to complex domain. Moreover it satisfies initial value problem

$$-(u')^{p-2}u'' - u^{p-1} = 0\,, \qquad u(0) = 0\,, \qquad u'(0) = 1$$

in the sense of ODE in complex domain. We also define function $\sinh_p$ as the unique solution of

$$-(u')^{p-2}u'' + u^{p-1} = 0\,, \qquad u(0) = 0\,, \qquad u'(0) = 1$$

and we try to find an analogy to classical formula

$$\sin(z) = -\mathrm{i}\sinh(\mathrm{i}z)\,.$$

Finally we will discus the remaining cases, p is an odd integer and p is not an integer.