$$\boxed{u'' = p(t)u \pm h(t)|u|^{\lambda}\operatorname{sgn} u(t); \quad u(0) = u(\omega),\ u'(0) = u'(\omega)} \qquad (1^{\pm})$$

- $\lambda > 1$
- $p, h\colon [0,\omega] \to \mathbb{R}$ are integrable functions, $h(t) \geq 0$ for a. e. $t \in [0,\omega]$

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$$\boxed{y'' + \delta y' + \alpha y + \beta y^3 = \gamma \sin t}$$

- $\alpha, \beta, \gamma \in \mathbb{R}, \quad \delta \geq 0$

G. Duffing, *Erzwungen Schwingungen bei veränderlicher Eigenfrequenz und ihre technisch Bedeutung*, Vieweg Heft 41/42, Vieweg, Braunschweig, 1918.

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$$\boxed{y'' + \alpha y + \beta y^3 = 0} \qquad (2)$$

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Equilibria:

▷ $y_1 = 0$

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Equilibria:

- ▷ $y_1 = 0$
- ▷ $\alpha\beta < 0 \quad \Rightarrow \quad y_{2,3} = \pm\sqrt{-\frac{\alpha}{\beta}}$

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Equilibria:

▷ $y_1 = 0$

▷ $\alpha\beta < 0 \quad \Rightarrow \quad y_{2,3} = \pm\sqrt{-\frac{\alpha}{\beta}}$ $\quad\nearrow\quad \alpha > 0,\ \beta < 0$ – y_1 stable, $y_{2,3}$ unstable

$$\boxed{u'' = p(t)u \pm h(t)|u|^{\lambda}\,\mathrm{sgn}\,u(t); \quad u(0) = u(\omega),\; u'(0) = u'(\omega)} \tag{$1^{\pm}$}$$

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G. Duffing, *Erzwungen Schwingungen bei veränderlicher Eigenfrequenz und ihre technisch Bedeutung*, Vieweg Heft 41/42, Vieweg, Braunschweig, 1918.

Equilibria:

- ▷ $y_1 = 0$
- ▷ $\alpha\beta < 0 \quad \Rightarrow \quad y_{2,3} = \pm\sqrt{-\frac{\alpha}{\beta}}$
  - ↗ $\alpha > 0,\ \beta < 0$ – y_1 stable, $y_{2,3}$ unstable
  - ↘ $\alpha < 0,\ \beta > 0$ – y_1 unstable, $y_{2,3}$ stable

$$\boxed{u'' = p(t)u + f(t); \quad u(0) = u(\omega), \; u'(0) = u'(\omega)}$$

- We say that $\boxed{p \in \mathcal{V}^+(\omega)}$ if

$$\left.\begin{array}{l} u \in AC^1([0,\omega]), \\ u''(t) \geq p(t)u(t) \quad \text{for a. e. } t \in [0,\omega], \\ u(0) = u(\omega), \; u'(0) = u'(\omega) \end{array}\right\} \implies \quad u(t) \geq 0 \quad \text{for } t \in [0,\omega].$$

Alternatively – Green's function is positive, or antimaximum principle holds

- We say that $\boxed{p \in \mathcal{V}^-(\omega)}$ if

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Alternatively – Green's function is negative, or maximum principle holds

- We say that $\boxed{p \in \mathcal{V}_0(\omega)}$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \; u'(0) = u'(\omega)$$

has a **positive** solution.

$$u'' = p(t)u + h(t)|u|^{\lambda} \operatorname{sgn} u(t); \quad u(0) = u(\omega), \; u'(0) = u'(\omega) \tag{1^+}$$

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad h \not\equiv 0 \tag{H_1}$$

Theorem A

Let (H_1) be fulfilled. Then

$$p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \Longrightarrow \quad (1^+) \text{ has only the trivial solution}$$

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au + bu^3} \tag{2}$$

- $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \iff a \geq 0, \qquad (H_1)$ holds $\iff b > 0$

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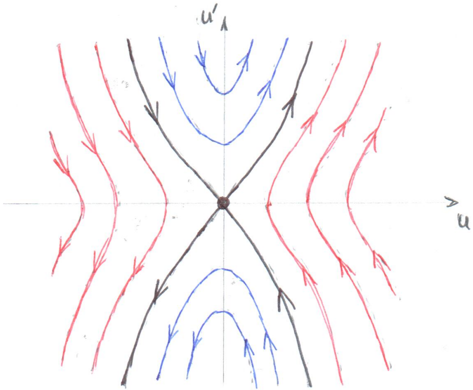

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$$h(t) > 0 \quad \text{for a. e. } t \in [0,\omega] \tag{H_2}$$

Theorem B

Let (H_2) be fulfilled. Then

$$p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \quad \Longrightarrow \quad (1^+) \text{ has a unique positive (resp. negative) solution}$$

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

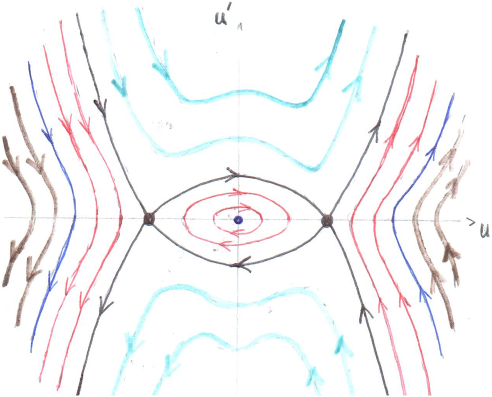
$$\boxed{u'' = au + bu^3} \tag{2}$$

- $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \iff a < 0, \qquad (H_2)$ holds $\iff b > 0$

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au + bu^3} \tag{2}$$

- $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \iff a < 0, \qquad (H_2)$ holds $\iff b > 0$
- $\boxed{a < 0,\ b > 0}$



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Let (H_1) be fulfilled. Then

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Theorem B

Let (H_2) be fulfilled. Then

$p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \implies (1^+)$ has a unique positive (resp. negative) solution

Theorem C

Let (H_1) be fulfilled. Then

$p \in \operatorname{Int} \mathcal{V}^+(\omega) \implies (1^+)$ has exactly three solutions (positive, negative, and trivial)

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au + bu^3} \tag{2}$$

- Theorem C claims: $-\frac{\pi^2}{\omega^2} < a < 0, \quad b > 0 \quad \Longrightarrow$ equation (2) has exactly three ω-periodic solutions (positive, negative, and trivial)

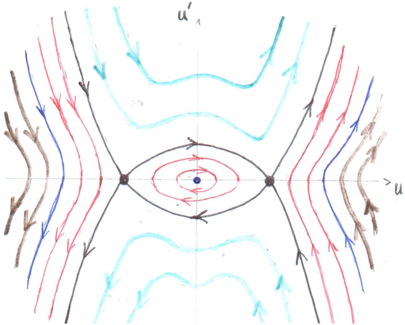
- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$
$$\boxed{u'' = au + bu^3} \tag{2}$$

- $-\frac{4\pi^2}{\omega^2} < a < 0, \quad b > 0 \quad \Longrightarrow$ equation (2) has exactly three ω-periodic solutions (positive, negative, and trivial) $\Longrightarrow$ every non-trivial ω-periodic solution to (2) is either positive or negative

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au + bu^3} \tag{2}$$

- $-\frac{4\pi^2}{\omega^2} < a < 0, \quad b > 0 \quad \Longrightarrow$ equation (2) has exactly three ω-periodic solutions (positive, negative, and trivial) $\Longrightarrow$ every non-trivial ω-periodic solution to (2) is either positive or negative



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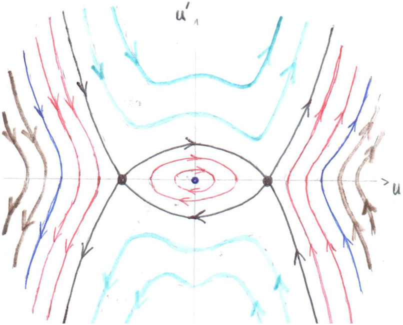

- u is a periodic solution of (2) corresponding to a closed orbit $\Longrightarrow$ the minimal period T of u satisfies

$$T \geq \frac{2\pi}{\sqrt{|a|}}$$

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u(t); \quad u(0) = u(\omega), \ u'(0) = u'(\omega) \tag{1^-}$$

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad h \not\equiv 0 \tag{H_1}$$

Theorem D

Let (H_1) be fulfilled. Then

(1^-) has a positive (resp. negative) solution $\iff p \in \mathcal{V}^-(\omega)$

$$u'' = p(t)u - h(t)|u|^{\lambda} \operatorname{sgn} u(t); \quad u(0) = u(\omega), \; u'(0) = u'(\omega) \tag{1^-}$$

$$h(t) \geq 0 \quad \text{for a. e. } t \in [0, \omega], \qquad h \not\equiv 0 \tag{H_1}$$

Theorem D

Let (H_1) be fulfilled. Then

(1^-) has a positive (resp. negative) solution $\iff p \in \mathcal{V}^-(\omega)$

It means that

$p \in \mathcal{V}^-(\omega) \implies$ (1^-) has a positive (resp. negative) solution

and

$p \notin \mathcal{V}^-(\omega) \implies$ (1^-) has no non-trivial sign-constant solution

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au - bu^3} \tag{2}$$

- $p \notin \mathcal{V}^-(\omega) \iff a \leq 0, \qquad (H_1)$ holds $\iff b > 0$

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au - bu^3} \tag{2}$$

- $p \notin \mathcal{V}^-(\omega) \iff a \leq 0, \qquad (H_1)$ holds $\iff b > 0$
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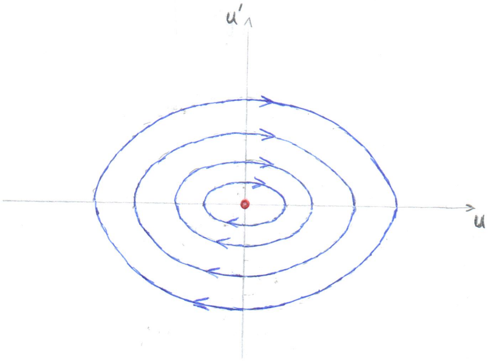


- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

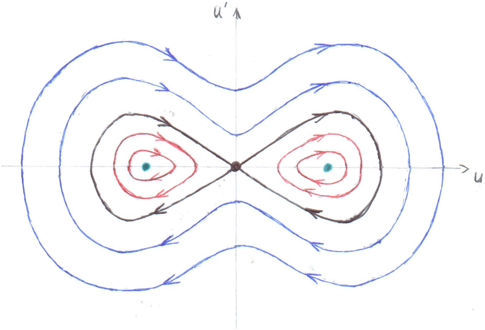
$$\boxed{u'' = au - bu^3} \tag{2}$$

- $p \in \mathcal{V}^-(\omega) \iff a > 0, \qquad (H_1)$ holds $\iff b > 0$

- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au - bu^3} \tag{2}$$

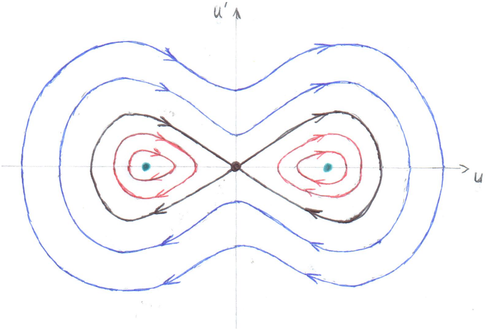
- $p \in \mathcal{V}^-(\omega) \iff a > 0, \qquad (H_1)$ holds $\iff b > 0$
- $\boxed{a > 0,\ b > 0}$



- $p(t) := a, \quad h(t) := b, \quad \lambda := 3$

$$\boxed{u'' = au - bu^3} \tag{2}$$

- $p \in \mathcal{V}^-(\omega) \iff a > 0, \qquad (H_1)$ holds $\iff b > 0$
- $\boxed{a > 0,\ b > 0}$



- $0 < a \leq \frac{2\pi^2}{\omega^2}, \quad b > 0 \implies$ equation (2) has a unique positive (resp. negative) ω-periodic solution

$$u'' = p(t)u \pm h(t)|u|^{\lambda} \operatorname{sgn} u$$

- $\lambda > 1$

$$u'' = p(t)u \pm h(t)|u|^\lambda \operatorname{sgn} u$$

- $\lambda > 1$

$$u'' = p(t)u \pm h(t)|u|^{\lambda-1}u$$

$$u'' = p(t)u \pm h(t)|u|^{\lambda} \operatorname{sgn} u$$

- $\lambda > 1$

$$u'' = p(t)u \pm h(t) \ln(1 + |u|)u$$

$$u'' = p(t)u \pm h(t)|u|^{\lambda} \operatorname{sgn} u$$

- $\lambda > 1$

$$u'' = p(t)u \pm h(t) \ln(1 + |u|)u$$

$$\rightsquigarrow$$

$$u'' = p(t)u \pm h(t)\varphi(u)u$$

- $\varphi \in C(\mathbb{R})$

$$u'' = p(t)u \pm h(t)|u|^{\lambda} \operatorname{sgn} u$$

- $\lambda > 1$

$$u'' = p(t)u \pm h(t) \ln(1 + |u|)u$$

$$\rightsquigarrow$$

$$u'' = p(t)u \pm h(t)\varphi(u)u$$

- $\varphi \in C(\mathbb{R})$

$$\rightsquigarrow$$

$$u'' = p(t)u \pm q(t, u)u$$

- $q\colon [0, \omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function