# On Asymptotic Classification of Solutions to The Singular Third- and Fourth-Order Emden–Fowler Equations

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Brno, January 26, 2015

Consider the equation

$$y^{(n)} = p(x, y, y', \dots, y^{(n-1)})|y|^k \operatorname{sign} y, \qquad (1)$$

$n \geq 2$, $k > 1$,

$p(x, y_0, \dots, y_{n-1}) > 0, \quad p \in C(\mathbf{R}^{n+1})$.

[*Kiguradze I. T., Chanturia T. A.* Asymptotic Properties of Solutions of Nonautonomous Ordinary Differential Equations. Kluver Academic Publishers, Dordreht-Boston-London. 1993.]
[*Kondratiev V. A., Samovol V. S.* On some asymptotic properties of solutions to Emden-Fowler type equation// Differ. equations. 1981. v. 17. N 4. p. 749–750.]

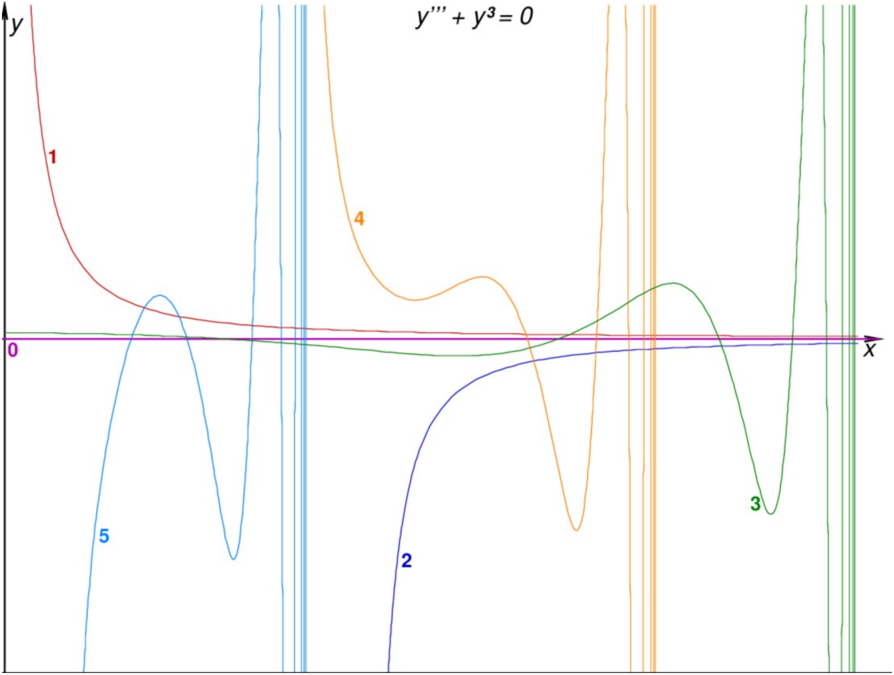
**The problem: to describe the asymptotic behavior of all possible maximally extended solutions to (1) in the case $p = p_0$ or $p = p(x)$.**

In the cases $n = 3$ and $n = 4$, $k > 1$ the asymptotic classification is obtained of all solutions to equation (1) (I.Astashova) for $p = p(x)$.

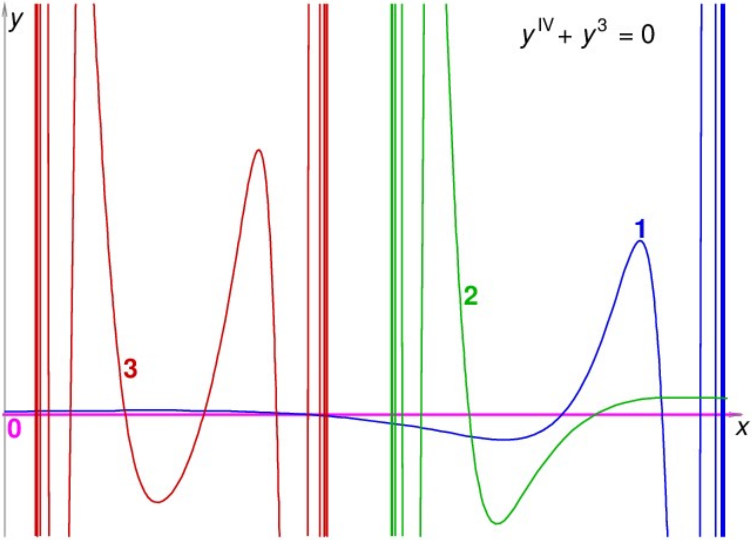
See

[*Astashova I. V.* Qualitative properties of solutions to quasilinear ordinary differential equations. In: Astashova I. V. (ed.) Qualitative Properties of Solutions to Differential Equations and Related Topics of Spectral Analysis: scientific edition, M.: UNITY-DANA, 2012, pp. 22–290. (Russian)]

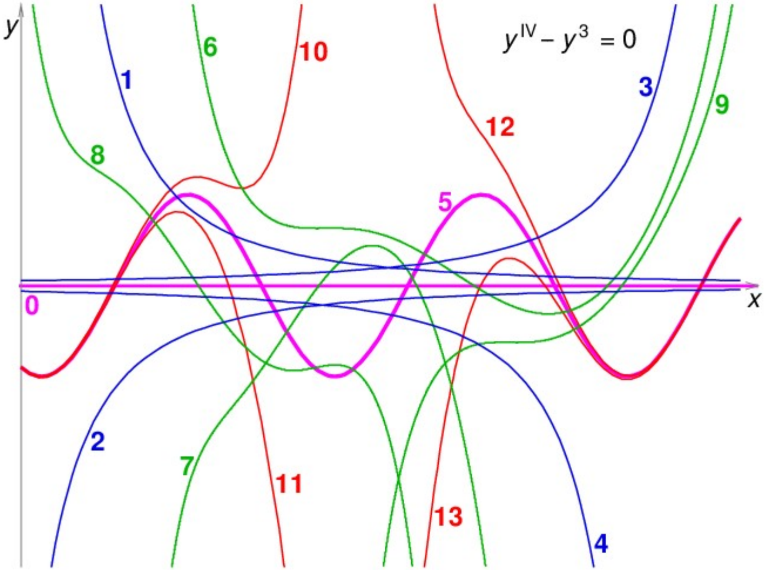
[*I. V. Astashova.* On Asymptotic Behavior of Solutions to a Forth Order Nonlinear Differential Equation. Proceedings of the 1st WSEAS International Conference on Pure Mathematics (PUMA '14), Tenerife, Spain, January 10-12, 2014, ISBN: 978-960-474-360-5, WSEAS Press,2013, pp. 32-41.]



$y''' + y^3 = 0$
y
x
0
1
2
3
4
5



y
$y^{IV} + y^3 = 0$
3
2
1
0
x



y
x
$y^{IV} - y^3 = 0$
0
1
2
3
4
5
6
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8
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11
12
13

For the equation

$$y^{(n)} + p_0 \, |y|^k \operatorname{sgn} y = 0, \quad n > 2, \quad k \in \mathbb{R}, \quad k > 1, \quad p_0 \neq 0, \quad (2)$$

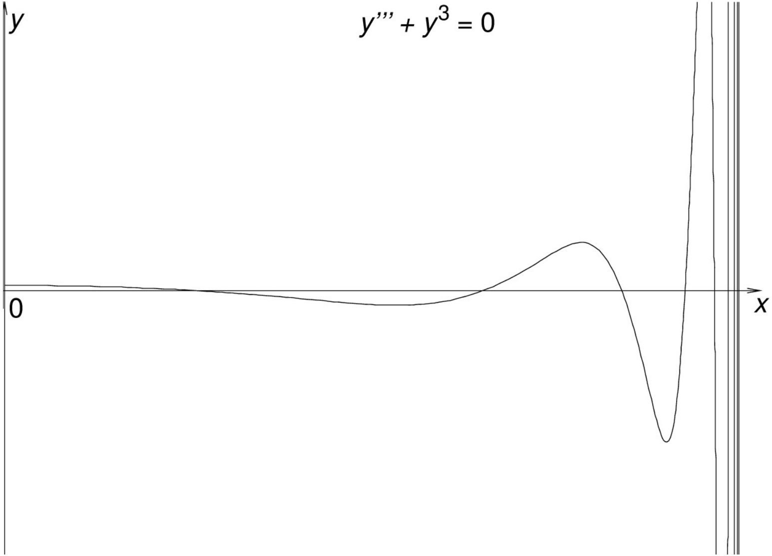
the following result is proved about the existence of oscillatory solutions with the same type of asymptotic behavior.

## Theorem

*For any integer $n > 2$ and real $k > 1$ there exists a non-constant oscillatory periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}} (x^* - x)^{-\alpha} h(\log(x^* - x)), \quad -\infty < x < x^*, \quad \alpha = \frac{n}{k-1}, \quad (3)$$

*is a solution to equation* (2).


y
$y''' + y^3 = 0$
0
x

### Theorem

*For any integer* $n > 2$ *and real positive* $k < 1$ *there exists a non-constant oscillatory periodic function* $h(s)$, *such that for any* $p_0$ *with* $(-1)^n p_0 > 0$ *and any real* $x^*$ *the function*

$$y(x) = |p_0|^{\frac{1}{k-1}} (x^* - x)^\gamma h(\log(x^* - x)), \qquad -\infty < x < x^*, \quad (4)$$

*is a solution to equation* (2) *with* $p(x, y_0, \ldots, y_{n-1}) = p_0$.

Consider the differential equation

$$y''' = p(x)|y|^k \operatorname{sgn} y, \quad 0 < k < 1, \tag{5}$$

with a globally defined positive continuous function $p(x)$ having positive limits $p_*$ and $p^*$ as $x \to \pm\infty$. Put $\beta = \frac{3}{1-k} > 0$.

### Theorem

*Any maximally extended solution to equation (5) is either*

*(i) the trivial solution* $y(x) \equiv 0$ *on* $(-\infty, +\infty)$, *or*

*(ii$_\pm$) a solution equal to zero on a semi-axis* $(-\infty,\, x^*]$ *and constant-sign with asymptotically power behavior on* $(x^*,\, +\infty)$, *namely*

$$y(x) = \pm C(p(x^*))\,(x - x^*)^\beta\,(1 + o(1)) \quad \textit{as}\ \ x \to x^* + 0, \qquad (6)$$

$$y(x) = \pm C(p^*)\,x^\beta\,(1 + o(1)) \quad \textit{as}\ \ x \to +\infty, \qquad (7)$$

*where* $C(p) = \left(\dfrac{(1-k)^3\,p}{3(k+2)(2k+1)}\right)^{\frac{1}{1-k}}$, *or* *(iii) a solution equal to zero on a semi-axis $[x_*,\ +\infty)$ and oscillating on $(-\infty,\ x_*)$ with its local extremum points $(x_j)_{j\in\mathbb{Z}}$ satisfying*

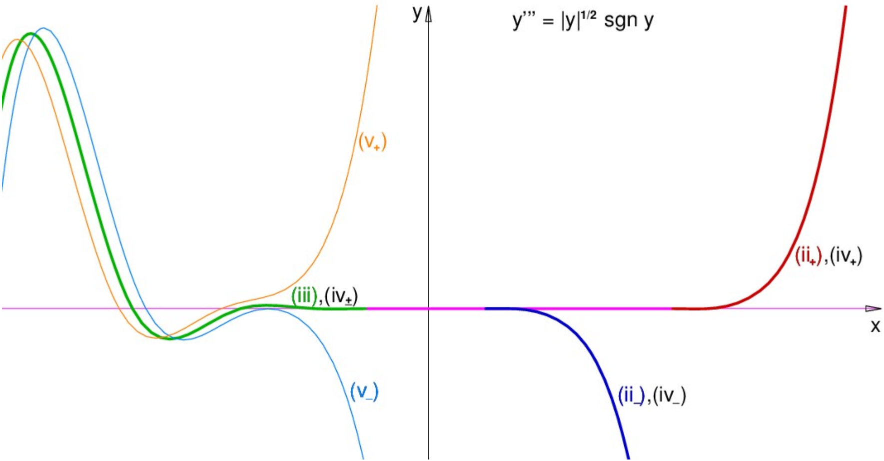
$$x_j \to -\infty, \qquad |y(x_j)| = |x_j|^{\beta+o(1)} \qquad \text{as } j \to -\infty, \quad (8)$$

$$x_j \to x_* - 0, \qquad |y(x_j)| = |x_* - x_j|^{\beta+o(1)} \qquad \text{as } j \to +\infty, \quad (9)$$

*or*

*(iv$_\pm$) a solution equal to zero on a segment* $[x_*,\ x^*]$ *(the case* $x_* = x^*$ *is admitted), oscillating on* $(-\infty, x_*)$ *with* (8)–(9) *satisfied, and constant-sign on* $(x^*, +\infty)$ *with* (6)–(7) *satisfied, or*

*($v_\pm$) a solution behaving as* (7) *at* $+\infty$, *as* (8) *at* $-\infty$, *and with no point* $x_0$ *satisfying* $y(x_0) = y'(x_0) = y''(x_0) = 0.$


y
y''' = |y|^{1/2} sgn y
(v+)
(ii+),(iv+)
(iii),(iv±)
x
(v−)
(ii−),(iv−)

# On Asymptotic Classification of Solutions to Emden–Fowler Singular Equations of the Fourth Order

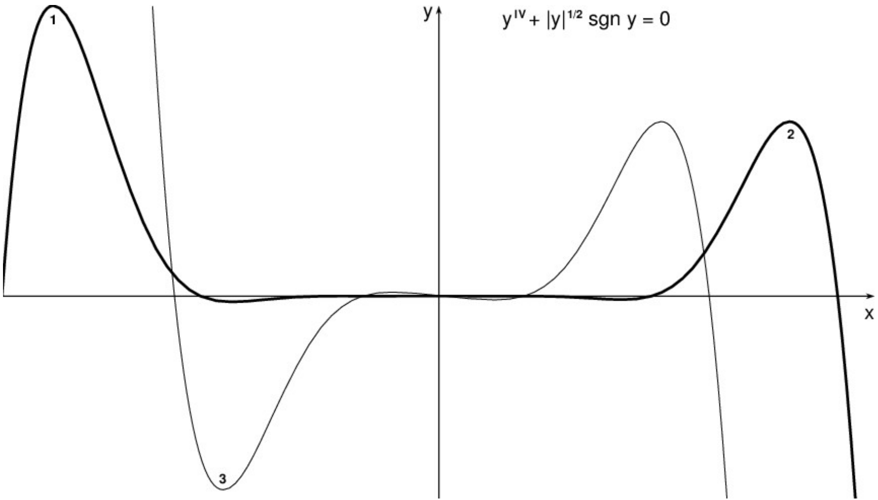
The asymptotic classification of all possible solutions to the fourth-order Emden–Fowler type differential equations

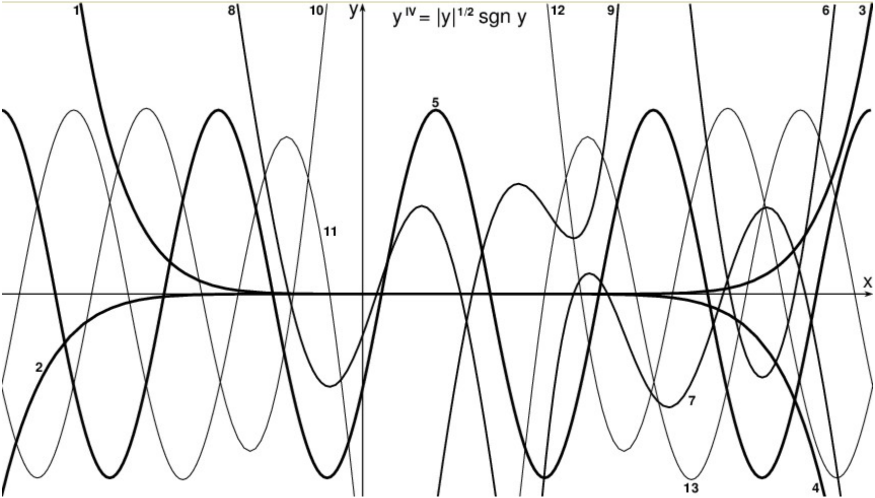
$$y^{\mathrm{IV}}(x) + p_0\,|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1,\ p_0 > 0 \tag{10}$$

and

$$y^{\mathrm{IV}}(x) - p_0\,|y|^k \operatorname{sgn} y = 0, \quad 0 < k < 1,\ p_0 > 0 \tag{11}$$

is given.


$y^{IV} + |y|^{1/2} \operatorname{sgn} y = 0$
y
x
1
2
3


$y^{IV} = |y|^{1/2} \operatorname{sgn} y$
y
x
1
2
3
4
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6
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9
10
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12
13

See

[*Astashova I.* On the existence of quasi-periodic solutions to Emden–Fowler type higher-order equations. Differential Equations, Maik Nauka/Interperiodica Publishing (Russian Federation), 2014, v. 50, no. 6, p. 847–848.]

[*Astashova I.* On quasi-periodic solutions to a higher-order Emden-Fowler type differential equation // Boundary Value Problems SpringerOpenJournal, 2014:174, p. 1–8. DOI:10.1186/s13661-014-0174-7]

[*Astashova I.* On the asymptotic behavior of solutions of differential equations with a singular nonlinearity // Differential Equations, Maik Nauka/Interperiodica Publishing (Russian Federation), 2014, v. 50, no. 11, p. 1551–1552.]

Thank you for your attention!