Positive Solutions of Periodic Type Boundary Value Problems for Singular in Phase Variables Differential Systems

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Let $-\infty < a < b < +\infty$,

$$\mathbb{R}_+ = [0,+\infty[\,, \quad \mathbb{R}^n_+ = \Big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 \geq 0, \dots, x_n \geq 0\Big\},$$
$$\mathbb{R}_{0+} =]0,+\infty[\,, \quad \mathbb{R}^n_{0+} = \Big\{(x_i)_{i=1}^n \in \mathbb{R}^n : \ x_1 > 0, \dots, x_n > 0\Big\},$$

and $f_i : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}$ $(i=1,\dots,n)$ and $\varphi_i : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i=1,\dots,n)$ are continuous functions. Consider the differential system

$$\frac{du_i}{dt} = f_i(t,u_1,\dots,u_n) \ \ (i=1,\dots,n) \tag{1}$$

with the boundary conditions

$$u_i(a) = \varphi_i(u_i(b)) \ \ (i=1,\dots,n). \tag{2}$$

A solution $(u_i)_{i=1}^n : [a,b] \to \mathbb{R}^n_{0+}$ of the system (1) satisfying the boundary conditions (2) is called a positive solution of the problem (1), (2).

The question on the existence of a positive solution of problems of the type (1), (2) has been investigated earlier mainly only for regular differential systems, i.e., for the systems whose right sides are continuous, or satisfy the local Carathéodory conditions on the set $[a,b] \times \mathbb{R}^n_+$ (see [1, 2] and the references therein).

Theorems below on the existence of a positive solution of the problem (1), (2) cover the cases in which the system under consideration has singularities in phase variables, in particular, the case where for arbitrary i and $k \in \{1,\dots,n\}$ the equality

$$\lim_{x_k \to 0} \big|f_i(t,x_1,\dots,x_n)\big| = +\infty \ \text{ for } \ x_j > 0 \ \ (j=1,\dots,n; \ \ j \neq k)$$

is fulfilled.

In Theorems 1 and 2 it is assumed, respectively, that the functions f_i $(i=1,\dots,n)$ and φ_i $(i=1,\dots,n)$ on the sets $[a,b] \times \mathbb{R}^n_{0+}$ and $\mathbb{R}_{0+}$ satisfy the inequalities

$$\sigma_i\big(f_i(t,x_1,\dots,x_n) - p_i(t)x_i\big) \geq q_i(t,x_i) \ \ (i=1,\dots,n), \tag{3}$$

$$q_i(t,x_i) \leq \sigma_i\big(f_i(t,x_1,\dots,x_n) - p_i(t)x_i\big) \leq \\ \leq \sum_{k=1}^n p_{ik}(t,x_1+\dots+x_n)x_k + q_0(t,x_1,\dots,x_n) \ \ (i=1,\dots,n), \tag{4}$$

and

$$\sigma_i(\varphi_i(x) - \alpha_i x) \geq 0, \quad \sigma_i(\varphi_i(x) - \beta_i x) \leq \beta_0 \ \ (i=1,\dots,n). \tag{5}$$

Here,

$$\sigma_i \in \{-1,1\}, \ \ \alpha_i > 0, \ \ \beta_i > 0, \ \ \sigma_i(\beta_i - \alpha_i) \geq 0 \ \ (i=1,\dots,n), \ \ \beta_0 \geq 0,$$

$p_i : [a,b] \to \mathbb{R}$ $(i=1,\dots,n)$ are continuous functions, $p_{ik} : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ and $q_i : [a,b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i,k=1,\dots,n)$ are nonincreasing in the second argument continuous functions, and

$q_0 : [a,b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a nonincreasing in the last n arguments continuous function. Moreover, p_i and q_i $(i = 1, \dots, n)$ satisfy the conditions

$$\sigma_i\biggl(\beta_i \exp\biggl(\int\limits_a^b p_i(s)\,ds\biggr) - 1\biggr) < 0 \quad (i = 1, \dots, n), \tag{6}$$

$$\max\bigl\{q_i(t,x) :\ a \le t \le b\bigr\} > 0 \ \text{ for } \ x > 0 \quad (i = 1, \dots, n). \tag{7}$$

Along with (1), (2) we consider the auxiliary problem

$$\frac{du_i}{dt} = (1-\lambda)\bigl(p_i(t)u_i + \sigma_i q_i(t,u_i)\bigr) + \lambda f_i(t, u_1, \dots, u_n) + \sigma_i \varepsilon \quad (i = 1, \dots, n), \tag{8}$$

$$u_i(a) = (1-\lambda)\beta_i u_i(b) + \lambda \varphi_i(u_i(b)) \quad (i = 1, \dots, n), \tag{9}$$

depending on the parameters $\lambda > 0$ and $\varepsilon > 0$.

Theorem 1 (**Principle of a priori boundedness**). *Let the inequalities* (3) *be fulfilled and let there exist positive constants ε_0 and ρ such that for arbitrary $\lambda \in [0,1]$ and $\varepsilon \in]0, \varepsilon_0]$ every positive solution $(u_i)_{i=1}^n$ of the problem* (8), (9) *admits the estimates*

$$u_i(t) < \rho \quad (i = 1, \dots, n).$$

Then the problem (1), (2) *has at least one positive solution.*

By $X = (x_{ik})_{i,k=1}^n$ we denote the $n \times n$ matrix with components $x_{ik} \in \mathbb{R}$ $(i, k = 1, \dots, n)$, and by $r(X)$ we denote the spectral radius of the matrix X. For any continuous function $p : [a,b] \to \mathbb{R}$ and positive number β satisfying the conditions

$$\Delta(p,\beta) = 1 - \beta \exp\biggl(\int\limits_a^b p(s)\,ds\biggr) \ne 0,$$

we put

$$g(p,\beta)(t,s) = \begin{cases} \dfrac{1}{\Delta(p,\beta)} \exp\biggl(\displaystyle\int\limits_s^t p(\tau)\,d\tau\biggr) & \text{for } \ a \le s \le t \le b, \\ \dfrac{\beta}{\Delta(p,\beta)} \exp\biggl(\displaystyle\int\limits_a^b p(\tau)\,d\tau + \int\limits_s^t p(\tau)\,d\tau\biggr) & \text{for } \ a \le t < s \le b \end{cases}$$

and

$$w(p,\beta)(t) = \frac{1}{\Delta(p,\beta)}\biggl[(1-\beta)\exp\biggl(\int\limits_a^t p(s)\,ds\biggr) + \beta \exp\biggl(\int\limits_a^b p(s)\,ds\biggr) - 1\biggr].$$

Theorem 2. *Let there exist continuous functions $\ell_i : [a,b] \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ such that along with* (4) *the inequality*

$$\lim_{x \to +\infty} r(H(x)) < 1 \tag{10}$$

be fulfilled, where $H(x) = (h_{ik}(x))_{i,k=1}^n$ and

$$h_{ik}(x) = \max\biggl\{\frac{1}{\ell_i(t)} \int\limits_a^b \bigl|g(p_i,\beta_i)(t,s)\bigr| p_{ik}(s,x)\ell_k(s)\,ds :\ a \le t \le b\biggr\} \quad (i, k = 1, \dots, n).$$

Then the problem (1), (2) *has at least one positive solution.*

This theorem can be proved on the basis of Theorem 1 and Theorem 3.1 of [3].

Now we pass to the case, where

$$|p_i(t_i)| = \max\big\{|p_i(s)| :\ a \le s \le b\big\} > 0, \quad p_i(t)p_i(t_i) \ge 0 \ \text{ for }\ a \le t \le b \ \ (i = 1, \dots, n) \tag{11}$$

and the inequalities (4) have the form

$$q_i(t, x_i) \le \sigma_i\big(f_i(t, x_1, \dots, x_n) - p_i(t)x_i\big) \le \\ \le |p_i(t)| \sum_{k=1}^{n} \frac{h_{ik}(x_1 + \dots + x_n)}{|w(p_k, \beta_k)(t)|}\, x_k + q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n), \tag{12}$$

where $h_{ik} : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i, k = 1, \dots, n)$ are continuous nonincreasing functions, and σ_i, q_i $(i = 1, \dots, n)$ and q_0 are the numbers and functions satisfying the above conditions.

From Theorem 2 it follows the following corollary.

Corollary 1. *If along with* (11) *and* (12) *the inequality* (10) *is fulfilled, where* $H(x) = (h_{ik}(x))_{i,k=1}^{n}$*, then the problem* (1), (2) *has at least one positive solution.*

As an example, we consider the problems

$$\frac{du_i}{dt} = \sigma_i\Big(\sum_{k=1}^{n} p_{ik}u_k + f_{0i}(t, u_1, \dots, u_n)\Big) \ \ (i = 1, \dots, n), \tag{13}$$

$$u_i(a) = u_i(b) \ \ (i = 1, \dots, n), \tag{14}$$

and

$$\frac{du_i}{dt} = \sigma_i\Bigg(\sum_{k=1}^{n} \frac{|1 - \beta_k| h_{ik}}{(1 - \beta_k)(t - a) + \beta_k(b - a)}\, u_k + f_{0i}(t, u_1, \dots, u_n)\Bigg) \ \ (i = 1, \dots, n), \tag{15}$$

$$u_i(a) = \beta_i u_i(b) \ \ (i = 1, \dots, n), \tag{16}$$

where $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, p_{ik} $(i, k = 1, \dots, n)$ and β_i $(i = 1, \dots, n)$ are the constants satisfying the inequalities

$$p_{ii} < 0, \ \ p_{ik} \ge 0 \ \ (i \ne k;\ \ i, k = 1, \dots, n), \tag{17}$$

$$\beta_i > 0, \ \ \sigma_i(\beta_i - 1) < 0 \ \ (i = 1, \dots, n), \tag{18}$$

h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants and $f_{0i} : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are continuous functions. Moreover, on the set $[a, b] \times \mathbb{R}^n_{0+}$ the inequalities

$$q_i(t, x_i) \le f_{0i}(t, x_1, \dots, x_n) \le q_0(t, x_1, \dots, x_n) \ \ (i = 1, \dots, n)$$

are fulfilled, where $q_0 : [a, b] \times \mathbb{R}^n_{0+} \to \mathbb{R}_+$ is a nonincreasing in the last n arguments continuous function and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are nonincreasing in the second argument continuous functions satisfying the conditions (7).

Corollary 2. *For the existence of at least one positive solution of the problem* (13), (14) *it is necessary and sufficient that real parts of the eigenvalues of the matrix* $(p_{ik})_{i,k=1}^{n}$ *be negative.*

Corollary 3. *For the existence of at least one positive solution of the problem* (15), (16) *it is necessary and sufficient that the matrix* $H = (h_{ik})_{i,k=1}^{n}$ *satisfy the inequality*

$$r(H) < 1. \tag{19}$$

Remark 1. In the conditions of Corollaries 2 and 3 the functions f_{0i} $(i = 1, \dots, n)$ may have singularities of arbitrary order in the least n arguments. For example, in (13) and (15) we may assume that

$$f_{0i}(t, x_1, \dots, x_n) = \sum_{k=1}^{n} q_{ik}(t) x_k^{-\mu_{ik}} \quad (i = 1, \dots, n),$$

where μ_{ik} $(i, k = 1, \dots, n)$ are positive constants and $q_{ik} : [a, b] \to \mathbb{R}_{0+}$ $(i, k = 1, \dots, n)$ are continuous functions.

The uniqueness of a positive solution of the problem (1), (2) can be proved only in the case where each function f_i has the singularity in the i-th phase variable only. More precisely, we consider the case when the system (1) has the following form

$$\frac{du_i}{dt} = p_i(t) u_i + \sigma_i \big(f_{0i}(t, u_1, \dots, u_n) + q_i(t, u_i) \big) \quad (i = 1, \dots, n). \tag{20}$$

Here $\sigma_i \in \{-1, 1\}$ $(i = 1, \dots, n)$, $p_i : [a, b] \to \mathbb{R}$ and $f_{0i} : [a, b] \times \mathbb{R}_+^n \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are continuous functions, and $q_i : [a, b] \times \mathbb{R}_{0+} \to \mathbb{R}_+$ $(i = 1, \dots, n)$ are nonincreasing in the second argument continuous functions. Moreover, p_i and q_i $(i = 1, \dots, n)$ satisfy the conditions (6) and (7).

Theorem 3. *Let on the sets $[a, b] \times \mathbb{R}_+^n$ and $\mathbb{R}_+$ the conditions*

$$\sigma_i \big(f_{0i}(t, x_1, \dots, x_n) - f_{0i}(t, y_1, \dots, y_n) \big) \operatorname{sgn}(x_i - y_i) \le \sum_{k=1}^{n} p_{ik}(t) |x_k - y_k| \quad (i = 1, \dots, n)$$

and

$$\sigma_i(\varphi_i(x) - \alpha_i x) \ge 0, \quad \sigma_i \Big[(\varphi_i(x) - \varphi_i(y)) \operatorname{sgn}(x - y) - \beta_i |x - y| \Big] \le 0 \quad (i = 1, \dots, n)$$

holds, where $p_{ik} : [a, b] \to \mathbb{R}_+$ $(i, k = 1, \dots, n)$ are continuous functions. Let, moreover, there exist continuous functions $\ell_i : [a, b] \to \mathbb{R}_{0+}$ $(i = 1, \dots, n)$ such that the matrix $H = (h_{ik})_{i,k=1}^n$ with the components

$$h_{ik} = \max \left\{ \frac{1}{\ell_i(t)} \int_a^b \big| g(p_i, \beta_i)(t, s) \big| p_{ik}(s) \ell_k(s) \, ds : \ a \le t \le b \right\} \quad (i, k = 1, \dots, n)$$

satisfies the inequality (19)*. Then the problem* (20), (2) *has a unique positive solution.*

The particular cases of (20) are the differential systems

$$\frac{du_i}{dt} = \sigma_i \bigg(\sum_{k=1}^{n} p_{ik} u_k + q_i(t, u_i) \bigg) \quad (i = 1, \dots, n) \tag{21}$$

and

$$\frac{du_i}{dt} = \sigma_i \bigg(\sum_{k=1}^{n} \frac{|1 - \beta_k| h_{ik}}{(1 - \beta_k)(t - a) + \beta_k(b - a)} \, u_k + q_i(t, u_i) \bigg) \quad (i = 1, \dots, n), \tag{22}$$

where p_{ik} and β_i $(i, k = 1, \dots, n)$ are the constants satisfying the inequalities (17) and (18), and h_{ik} $(i, k = 1, \dots, n)$ are nonnegative constants.

Theorem 3 results in the following corollaries.

Corollary 4. *For the existence of a unique positive solution of the problem* (21), (14) *it is necessary and sufficient that real parts of eigenvalues of the matrix $(p_{ik})_{i,k=1}^n$ be negative.*

Corollary 5. *For the existence of a unique positive solution of the problem* (22), (16) *it is necessary and sufficient that the matrix* $H = (h_{ik})_{i,k=1}^{n}$ *satisfy the inequality* (19).

Remark 2. In the conditions of Theorem 3 and its corollaries, the functions q_i $(i = 1, \dots, n)$ may have singularities of arbitrary order in the second argument. For example, in (20), (21) and (22) we may assume that

$$q_i(t, x) = q_{i1}(t)x^{-\mu_{i1}} + q_{i2}(t)\exp(x^{-\mu_{i2}}) \ \ (i = 1, \dots, n),$$

where $\mu_{i1} > 0$, $\mu_{i2} > 0$ $(i = 1, \dots, n)$, and $q_{ik} : [a, b] \to \mathbb{R}_+$ $(i = 1, \dots, n;\ k = 1, 2)$ are continuous functions such that

$$\max\big\{q_{i1}(t) + q_{i2}(t) : \ a \le t \le b\big\} > 0 \ \ (i = 1, \dots, n).$$

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