Periodic and quasi-periodic motions of a relativistic particle under a central force field with applications to scalar boundary periodic problems

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Mathematical objective

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Consider the radially symetrical systems with singularity

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \qquad (1)$$

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Solution: A C^2 function $x : \mathbb{R} \to \mathbb{R}^2$ such that

$$x(t) \neq 0, \qquad |\dot{x}(t)| < 1, \qquad t \in \mathbb{R}.$$

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$T - rad.$ periodic solutions: is a solution $x(t) = r(t)e^{i\theta(t)}$ such that

$$r(t) = r(t+T), \qquad \omega := \frac{\theta(t+T) - \theta(t)}{T} \quad \forall \quad t \in \mathbb{R}.$$

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Type of $T-rad.$ periodic solutions:

- $T-$periodic solutions: $\omega \in \frac{2\pi}{T}\cdot\mathbb{Z}$
- subharmonic solutions: $\omega \in \frac{2\pi}{T}\cdot\mathbb{Q}$
- quasi-periodic solutions: $\omega \in \frac{2\pi}{T}\cdot\mathbb{R}\setminus\mathbb{Q}$

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`As consequence of radially symmetrical systems with singularity` we can study the existence of $T - \textit{periodic}$ solutions for the scalar equation

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \tag{2}$$

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$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \qquad (2)$$

`Definition of T−periodic solutions:` A C^2 function $r:\mathbb{R} \longrightarrow \mathbb{R}$ verifying

$$r(t+T) = r(t) \qquad \forall \quad t \in \mathbb{R}.$$

Mathematical objective

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \tag{2}$$

Important particular equations:

- **Gylden-Meshcherskii equation:** (1) with $f(t,x) := M(t)/x^3$.
- **Lazer and Solimini equation with weak singularity:** (2) with $f(t,r) := 1/r^\gamma + e(t)$.
- **Brillouin beam focusing equation:** (2) with $f(t,x) := -b(1+\cos t)x + 1/x^\gamma$.
- **Equations with mixed singularity:** (2) with $f(t,x) := n(t)/x^\gamma + e(t)$.

Physical motivation of eq. (1)

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The motion of a particle subjected to the influence of an (autonomous) central force field in the plane may be mathematically modelled by

$$\ddot{x} = f(x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\},$$

and it has had a great importance in Mechanics at the beginning of the seventeenth century.
However many problems in Celestial Mechanics involve non-autonomous central force field in the plane. The best known is the Gylden-Meshcherskii problem

$$\ddot{x} = -M(t)\frac{x}{|x|^3},$$

where $M(t) = G(m_1(t) + m_2(t))$, G is the gravitational constant.

Physical motivation of eq. (1)

Gylden-Meshcherskii problem

$$\ddot{x} = -M(t)\frac{x}{|x|^3}$$

was proposed to explain the secular acceleration in the Moon's longitude, but nowadays it is used to describes many phenomena including: *the evolution of binary stars, dynamic of particles around pulsating starts a many particle and many others.*

Physical motivation of eq. (1)

When dealing with particles moving at speed close to that of light may be important to take into account the relativistic effects. As result when the speed of light is normalized to one, we are led to consider the following family of second-order systems in the plane:

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\},$$

and the Gylden-Mashcherskii problem becomes

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = -M(t)\frac{x}{|x|^3}, \qquad x \in \mathbb{R}^2 \setminus \{0\}.$$

Some remarks on eq. (1)

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$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \qquad (1)$$

Remark:

If our force field is globally repulsive, i.e.,

$$f(t,r) > 0 \qquad \forall \quad (t,r) \in \mathbb{R} \times (0,\infty),$$

then (1) has no $T - rad.$ periodic solutions.

Some remarks on eq. (1)

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \qquad (1)$$

Remark:

If our force field is attractive at some level $r_* > 0$ and autonomous, i.e.,

$$f(r_*) < 0,$$

then there are $T - rad$. periodic solutions of (1) with constant angular velocity equal to

$$|\omega| = \frac{\sqrt{2}}{r_*\sqrt{1+\sqrt{1+\left(\frac{2}{r_* f(r_*)}\right)^2}}}.$$

Main result

Main result

The radial symmetry of equation

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\} \qquad (1)$$

con be exploited to reduce its order, by introducing polar coordinates $x(t) = r(t)e^{i\theta(t)}$. As result we obtain first-order (Hamiltonian) system defined on $\{r > 0\}$ and depending on the parameter μ:

$$\dot{r} = \frac{rp}{\sqrt{\mu^2 + r^2 + r^2p^2}}, \quad \dot{p} = \frac{\mu^2}{r^2\sqrt{\mu^2 + r^2 + r^2p^2}} + f(t,r). \quad (3)$$

$\mu := r^2\dot{\theta}/\sqrt{1-\dot{r}^2 - r^2\dot{\theta}^2}$ and $p := \dot{r}/\sqrt{1-\dot{r}^2-r^2\dot{\theta}^2}$ are *relativistic angular and linear momentum*, respectively.

Main result

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\} \tag{1}$$

$$\dot{r} = \frac{rp}{\sqrt{\mu^2 + r^2 + r^2p^2}}, \quad \dot{p} = \frac{\mu^2}{r^2\sqrt{\mu^2 + r^2 + r^2p^2}} + f(t,r). \tag{3}$$

If $(r, p; \mu)$ is a solution of (3), then, taking θ any primitive of $\mu/r(t)\sqrt{\mu^2 + r^2(t) + r^2(t)p^2(t)}$, $x(t) = r(t)e^{i\theta(t)}$ is a solution of (1) with angular momentum μ.

Remark

x is $T-rad.$ periodic *if and only if* r and p are both $T-$periodic, and the rotacional number can be computed by

$$\text{rot}(r, p; \mu) = \frac{\mu}{T}\int_0^T \frac{dt}{r(t)\sqrt{\mu^2 + r^2(t) + r^2(t)p^2(t)}}.$$

Main result

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\} \tag{1}$$

Torres, Ureña and Zamora proved the following statements:

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If $f(t, r_*) < 0$ for all $t \in \mathbb{R}$ then there exists some $T - rad.$ periodic solution x_* of (1) with $\min_{t\in\mathbb{R}} |x(t)| = r_*$.

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Under the previous assumptions there exists $\omega_* > 0$ such that: for any $\omega \in (-\omega_*, \omega_*) \setminus \{0\}$ there is some $T - rad.$ periodic solution x_ω of (1) with $\min_{t\in\mathbb{R}} |x_\omega(t)| \geq r_*$ and $\text{rot}(x_\omega) = \omega$.

Main result

$$\dot{r} = \frac{rp}{\sqrt{\mu^2 + r^2 + r^2p^2}}, \quad \dot{p} = \frac{\mu^2}{r^2\sqrt{\mu^2 + r^2 + r^2p^2}} + f(t, r). \quad (3)$$

We introduce a new parameter ($\lambda > 0$) by considering the change of variable:

$$r(t) = \lambda(1 + \widetilde{r}(t)), \quad p(t) = \lambda\widetilde{p}(t),$$

where $(\widetilde{r}, \widetilde{p}) \in C_0 \times C_T$.
According to the new variables (3) can be rewritten as

$$\dot{\widetilde{r}} = N_1[\lambda, \widetilde{r}, \widetilde{p}; \mu], \qquad \dot{\widetilde{p}} = N_2[\lambda, \widetilde{r}, \widetilde{p}; \mu], \qquad (4)$$

where $N_i : \Omega \subset \mathbb{R} \times Y$ where
$\Omega = \{(\lambda, \widetilde{r}, \widetilde{p}, \mu) \in \mathbb{R} \times Y : \lambda > 0, \min_{t\in\mathbb{R}} \widetilde{r}(t) > -1\}$ and
$Y = C_0 \times C \times \mathbb{R}$.

Main result

We can write our system of differential equations

$$\dot{\widetilde{r}} = N_1[\lambda, \widetilde{r}, \widetilde{p}; \mu], \qquad \dot{\widetilde{p}} = N_2[\lambda, \widetilde{r}, \widetilde{p}; \mu], \tag{4}$$

as fixed point problem (depending on a parameter) defined on suitable open set of Y

$$y = F[\lambda; y], \tag{5}$$

where $F : \Omega \to Y$ is a completely continuous operator.
If there exists $r_* > 0$ such that

$$\int_0^T \max_{r \in [\lambda, \lambda + T/2]} f(t, r) dt < 0 \qquad \forall \quad \lambda \geq r_*,$$

then there exists U_λ such that $\deg_{LS}\left(I - F[\cdot; 0], U_\lambda, 0\right) \neq 0$.

Main result

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \qquad (1)$$

Theorem 1

Assume the existence of $r_* > 0$ such that

$$\int_0^T \max_{r \in [\lambda, \lambda + T/2]} f(t,r)dt < 0 \qquad \forall \quad \lambda \geq r_*.$$

Then either

$$\left\{ \min_{t \in \mathbb{R}} |x(t)| : x \text{ is a } T\text{−rad. periodic solution of } (1) \right\} = (0, \infty)$$

or there exist $T - rad.$ periodic solutions of (1) with angular velocity equal to 0.

Main result

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

According to the definition

$$\text{rot}(r,p;\mu) := \frac{\mu}{T}\int_0^T \frac{dt}{r(t)\sqrt{\mu^2 + r^2(t) + r^2(t)p^2(t)}},$$

we obtain

Theorem 2

Under the previous assumption there exists $\omega_* > 0$ with the following property: for every $\omega \in (-\omega_*, \omega_*) \setminus \{0\}$ there is a $T-$rad. periodic solution $x_\omega = x_\omega(t)$ of (1) such that $\text{rot}(x_\omega) = \omega$.

Main result

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \tag{1}$$

Finally, we point out that the above Theorems also can become false if our particle is restricted to be on a line instead of on the plane. In this case problem (1) is reduced to

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \tag{2}$$

This implies, according to Theorem 1, that under the global attractiveness of f, for every positive number r there exists a $T-$rad. periodic solution of (1) x_r such that $\min_{t\in\mathbb{R}} |x_r(t)| = r$.

Main result

On the contrary, if one would know that there exists some level $r_0 > 0$ such that

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = f(t,x)\frac{x}{|x|}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \qquad (1)$$

has no $T-$rad. periodic solutions then Theorem 1 provides the existence of $T-$periodic solutions of

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \qquad (2)$$

Assuming the existence of a level r_0 such that

$$\int_0^T \min_{r\in[r_0,r_0+T/2]} f(t,r)dt > 0,$$

the previous statement holds.

Main result

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = f(t,r), \qquad r > 0. \tag{2}$$

Theorem 3

Under the assumption of Theorem 1. If there exists at level r_0 such that

$$\int_0^T \min_{r\in[r_0,r_0+T/2]} f(t,r)dt > 0,$$

then (2) has at least one (positive) $T-$periodic solution.

Important equations: The Gylden-Meshcherskii equation

$$\frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{1-|\dot{x}|^2}}\right) = M(t)\frac{x}{|x|^3}, \qquad x \in \mathbb{R}^2 \setminus \{0\}. \qquad (6)$$

According to the results of P. J. Torres, A. J. Ureña and M. Zamora in *Bull. Lond. Math. Soc.* one can prove the existence of $T - rad.$ periodic solutions of (6) when $M < 0$.

Corollary 1

Assume that $\overline{M} < 0$. Then there exists at least one $T - rad.$ periodic of (6). In fact, $\overline{M} < 0$ is a necessary and sufficient condition for the existence of non-constant $T - rad.$ periodic solution of (6).

Important equations: The Lazer and Solimini equation

Repulsive case:

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right)=\frac{1}{r^\gamma}+e(t), \qquad r>0. \tag{7}$$

When we deal with strong singularity $\gamma \geq 1$, according to the results of C. Bereanu, D. Gheorghe and M. Zamora in *Communication in Contempary Mathematics 2013*, we are able to deduce the existence of a $T-$periodic solution of (7) whenever there are α and β positive lower and upper functions.

Theorem (C. Bereanu, D. Gheorghe and M. Zamora, 2013)

Assuming $\gamma \geq 1$, if $\overline{e} < 0$ then there exist at least one $T-$periodic solution of (7).

Repulsive case:

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = \frac{1}{r^\gamma} + e(t), \qquad r > 0. \tag{7}$$

When we deal with weak singularity $0 < \gamma < 1$, in the line of the original paper of Lazer and Solimini we proved

Theorem (C. Bereanu, D. Gheorghe and M. Zamora, 2013)

If $0 < \gamma < 1$ there exists a continuous function $e : [0, T] \to \mathbb{R}$ with $\overline{e} < 0$ such that (7) has no $T-$periodic solutions.

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THERE WERE NO RESULTS ON THE EXISTENCE OF $T-$PERIODIC SOLUTIONS OF (7) IN THE WEAK CASE.

Repulsive case:

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = \frac{1}{r^\gamma} + e(t), \qquad r > 0. \tag{7}$$

According to our main Theorem 3 we can prove

Corollary

Assume $0 < \gamma < 1$. If

$$-\left(\frac{2}{T}\right)^\gamma < \bar{e} < 0,$$

then there exists at least one $T-$periodic solution of (7).

Important equations: The Lazer and Solimini equation

Attractive case:

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = -\frac{1}{r^\gamma} + e(t), \qquad r > 0. \tag{8}$$

Theorem (C. Bereanu, D. Gheorghe and M. Zamora, 2013)

Assume $\gamma > 0$. There exists at least one positive $T-$periodic solution of (8) if and only if $\overline{e} > 0$.

Important equations: The Brillouin beam focusing equation

$$\frac{d}{dt}\left(\frac{\dot{r}}{\sqrt{1-\dot{r}^2}}\right) = -b(1+\cos t)r + \frac{1}{r^\gamma}, \qquad r > 0. \qquad (9)$$

Strong case: Published in CCM.

Theorem (C. Bereanu, D. Gheorghe and M. Zamora, 2013)

Assume $\gamma \geq 1$. Then there exists at least one $T-$periodic solution of (9) for all $b > 0$.

Weak case: Published in NoDea.

Theorem (C. Bereanu, D. Gheorghe and M. Zamora, 2013)

Assume $0 < \gamma < 1$. If

$$b < \left(\frac{2}{T}\right)^{\gamma+1},$$

then there exists at least one $T-$periodic solution of (9).

Important equations: Mixed singularities

Important equations: Mixed singularities

$$\frac{d}{dt}\left(\frac{\dot r}{\sqrt{1-\dot r^2}}\right)=\frac{n(t)}{r^\gamma}+e(t), \qquad r>0. \tag{10}$$

Corollary

Assume $E<0$. If $N_+>N_-$ and

$$-2^\gamma N_-\left[\left(-1+\frac{1}{1-\left(\frac{N_-}{N_+}\right)^{\frac{1}{1+\gamma}}}\right)T\right]^{-\gamma}+N_+\left(\frac{T}{2-2\left(\frac{N_-}{N_+}\right)^{\frac{1}{1+\gamma}}}\right)^{-\gamma}+E>0,$$

then there exists at least one $T-$periodic solution of (10).

Important equations: Mixed singularities

$$\frac{d}{dt}\left(\frac{\dot r}{\sqrt{1-\dot r^2}}\right)=\frac{n(t)}{r^\gamma}+e(t),\qquad r>0. \tag{10}$$

Corollary

Assume $E>0$. If $N_->N_+$ and

$$-2^\gamma N_+\left[\left(-1+\frac{1}{1-\left(\frac{N_+}{N_-}\right)^{\frac{1}{1+\gamma}}}\right)T\right]^{-\gamma}+N_-\left(\frac{T}{2-2\left(\frac{N_+}{N_-}\right)^{\frac{1}{1+\gamma}}}\right)^{-\gamma}+E<0,$$

then there exists at least one $T-$periodic solution of (10).

Reference

Reference

The results presented here were published in

(M. Zamora, 2013)

New periodic and quasi-periodic motions of a relativistic particle under a planar central force field with applications to scalar boundary periodic problems, **Electronic J. Qualitative Theory of Differential Equations** 31, 1-16 (2013)

Thank you for your attention.