Nonlinear Nonlocal Boundary Value Problems for Singular in a Phase Variable Second Order Differential Equations

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Main Results

$$u'' = f(t, u) \tag{1.1}$$

$$u(a) = \varphi_1(u), \quad u(b) = \varphi_2(u) \tag{1.2}$$

$$u(a) = \varphi_1(u), \quad u'(b) = \varphi_2(u) \tag{1.3}$$

$$f :]a, b[\times]0, +\infty[\to \mathbb{R}_+, \qquad \varphi_i : C([a, b]; \mathbb{R}_+) \to \mathbb{R}_+ \ \ (i = 1, 2)$$

Positive solution: $u \in C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$,

$$u(t) > 0 \ \text{ for } \ a < t < b$$

$$p_0(t,x) \le f(t,x) \le p_1(t,x) + p_2(t,x)(1+x) \tag{1.4}$$

$$\varphi_i(u) \le r\|u\| + r_0 \ \ (i = 1,2); \tag{1.5}$$

$$\varphi_1(u) \le r\|u\| + r_0, \quad \varphi_2(u) \le \|u\|_{[a,b_0]} \tag{1.6}$$

$$\varphi_1(u) + (b-a)\varphi_2(u) \le r\|u\| + r_0, \tag{1.7}$$

$p_i :]a,b[\, \times\,]0,+\infty[\, \to \mathbb{R}_+ \ (i = 0,1,2), \ b_0 \in\,]a,b[$

$$\|u\|_{[a,b_0]} = \max\big\{u(t) : \ a \le t \le b_0\big\}.$$

We are, mainly, interested in the case, in which

$$\lim_{x \to 0} p_i(t, x) = +\infty \text{ for } t \in I \quad (i = 0, 1, 2),$$

where $I \subset [a, b]$ is a set of positive measure. In this case,

$$\lim_{x \to 0} f(t, x) = -\infty \text{ for } t \in I,$$

that is, the equation (1.1) is singular with respect to the phase variable.

$$u'' = -\frac{t^2}{32u^2}; \quad u(a) = 0, \;\; u(b) = 0$$

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$$u'' = -\frac{1-t}{u}\,; \quad u(a)=0, \ \ u(b)=0$$

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$$u'' = \frac{h(t)}{u^{\lambda}}\,; \quad u(a) = 0, \;\; u(b) = 0$$

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$$u'' = f(t,u) \tag{1.1}$$

$$u(a) = \varphi_1(u), \quad u(b) = \varphi_2(u) \tag{1.2}$$

$$p_0(t,x) \le f(t,x) \le p_1(t,x) + p_2(t,x)(1+x) \tag{1.4}$$

$$\varphi_i(u) \le r\|u\| + r_0 \ \ (i=1,2) \tag{1.5}$$

Theorem 1. *If along with* (1.4) *and* (1.5) *the conditions*

$$0 < \int_a^b (t-a)(b-t)p_i(t,x)\,dt < +\infty \ \ \textit{for} \ \ x > 0 \ \ (i = 0,1),$$

$$r < 1, \quad \lim_{x \to 0} \int_a^b (t-a)(b-t)p_2(t,x)\,dt < (1-r)(b-a)$$

are fulfilled, then the problem (1.1), (1.2) *has at least one positive solution.*

$$u'' = f(t,u) \tag{1.1}$$

$$u(a) = \varphi_1(u), \quad u(b) = \varphi_2(u) \tag{1.2}$$

$$p_0(t,x) \le f(t,x) \le p_1(t,x) + p_2(t,x)(1+x) \tag{1.4}$$

$$\varphi_1(u) \le r\|u\| + r_0, \quad \varphi_2(u) \le \|u\|_{[a,b_0]} \tag{1.6}$$

Theorem 2. *If along with* (1.4), (1.6) *and the conditions*

$$0 < \int_a^b (t-a)(b-t)p_i(t,x)\,dt < +\infty \ \ \textit{for} \ \ x > 0 \ \ (i = 0,1),$$

$$r < 1, \quad \lim_{x \to +\infty} \int_a^b (t-a)(b-t)p_2(t,x)\,dt < (1-r)(b-b_0)$$

are fulfilled, then the problem (1.1), (1.2) *has at least one positive solution.*

$$u'' = f(t,u) \tag{1.1}$$

$$u(a) = \varphi_1(u), \quad u'(b) = \varphi_2(u) \tag{1.3}$$

$$p_0(t,x) \le f(t,x) \le p_1(t,x) + p_2(t,x)(1+x) \tag{1.4}$$

$$\varphi_1(u) + (b-a)\varphi_2(u) \le r\|u\| + r_0 \tag{1.7}$$

Theorem 3. *If along with* (1.4) *and* (1.7) *the conditions*

$$0 < \int_a^b (t-a)p_i(t,x)\,dt < +\infty \ \ \textit{for} \ \ x > 0 \ \ (i = 0,1),$$

$$r < 1, \quad \lim_{x\to+\infty} \int_a^b (t-a)p_2(t,x)\,dt < (1-r)(b-a)$$

are fulfilled, then the problem (1.1), (1.3) *has at least one positive solution.*

Note that if the conditions of Theorems 1 or 2 (of Theorem 3) are fulfilled, but

$$\int_a^b p_i(t,x)\,dt = +\infty \ \text{ for } \ x > 0 \ \ (i = 0, 1),$$

then the equation (1.1) has a nonintegrable singularity in a time variable at the point $t = a$ or $t = b$ (at the point $t = a$).

Example:

$$u'' = \sum_{k=1}^{n} \frac{f_k(t)}{q_k(u)}\, u^{\lambda_k} + \frac{f_0(t)}{q_0(u)} \tag{1.8}$$

$$u(a) = \int_a^b h_1(u(s))\, d\ell_1(s), \quad u(b) = \int_a^b h_2(u(s))\, d\ell_2(s); \tag{1.9}$$

$$u(a) = \int_a^b h_1(u(s))\, d\ell_1(s), \quad u(b) = \int_a^{b_0} h_2(u(s))\, d\ell_0(s); \tag{1.10}$$

$$u(a) = \int_a^b h_1(u(s))\, d\ell_1(s), \quad u'(b) = \int_a^b h_2(u(s))\, d\ell_1(s); \tag{1.11}$$

$$0 < \lambda_k < 1 \ \ (i = 1, \dots, n-1), \ \ \lambda_n = 1, \ \ b_0 \in]a, b[\,,$$

$f_k :]a, b[\to \mathbb{R}_+$, $q :]0, +\infty[\to]0, +\infty[\,$, $\lim\limits_{x\to 0} q_k(x) = 0$, $h_i : \mathbb{R}_+ \to \mathbb{R}_+$
$\ell_i(b) - \ell_i(a) = 1$ $(i = 1, 2)$, $\ell_0(b_0) - \ell_0(a) = 1$

$$u'' = \sum_{k=1}^{n} \frac{f_k(t)}{q_k(u)}\, u^{\lambda_k} + \frac{f_0(t)}{q_0(u)} \tag{1.8}$$

$$u(a) = \int_a^b h_1(u(s))\, d\ell_1(s), \quad u(b) = \int_a^b h_2(u(s))\, d\ell_2(s) \tag{1.9}$$

Corollary 1. *If*

$$0 < \int_a^b (t-a)(b-t) f_0(t)\, dt < +\infty, \int_a^b (t-a)(b-t) f_i(t)\, dt < +\infty \ (i = 1, \dots, n),$$
$$\lim_{x \to +\infty} \frac{h_i(x)}{x} < 1 \ \ (i = 1, 2), \quad \lim_{x \to +\infty} q_n(x) = +\infty,$$

then the problem (1.8), (1.9) *has at least one positive solution.*

$$u'' = \sum_{k=1}^{n} \frac{f_k(t)}{q_k(u)}\, u^{\lambda_k} + \frac{f_0(t)}{q_0(u)} \tag{1.8}$$

$$u(a) = \int_a^b h_1(u(s))\, d\ell_1(s), \quad u(b) = \int_a^{b_0} h_2(u(s))\, d\ell_0(s) \tag{1.10}$$

Corollary 2. *If*

$$0<\int_a^b (t-a)(b-t)f_0(t)\, dt<+\infty, \int_a^b (t-a)(b-t)f_k(t)\, dt<+\infty \ (i=1,\dots,n),$$

$$\lim_{x\to+\infty} \frac{h_1(x)}{x} < 1, \quad h_2(x) \le x \ \text{ for } \ x>0, \quad \lim_{x\to+\infty} q_n(x) = +\infty,$$

then the problem (1.8), (1.10) *has at least one positive solution.*

$$u'' = \sum_{k=1}^{n} \frac{f_k(t)}{q_k(u)}\, u^{\lambda_k} + \frac{f_0(t)}{q_0(u)} \tag{1.8}$$

$$u(a) = \int_a^b h_1(u(s))\, d\ell_1(s), \quad u'(b) = \int_a^b h_2(u(s))\, d\ell_2(s) \tag{1.11}$$

Corollary 3. *If*

$$0 < \int_a^b (t-a) f_0(t)\, dt < +\infty, \quad \int_a^b (t-a) f_k(t)\, dt < +\infty \ (k=1,\dots,n),$$
$$\limsup_{x\to+\infty} \frac{h_1(x)}{x} + (b-a) \limsup_{x\to+\infty} \frac{h_2(x)}{x} < 1, \quad \lim_{x\to+\infty} q_n(x) = +\infty,$$

then the problem (1.8), (1.11) *has at least one positive solution.*

Auxiliary Propositions

$$p_0(t,u(t)) \le -u''(t) \le p_1(t,u(t)) + p_2(t,u(t))(1+u(t)) \quad (2.1)$$

$$u(a) \le r\|u\| + r_0, \quad u(b) \le r\|u\| + r_0 \quad (2.2)$$

$$u(a) \le r\|u\| + r_0, \quad u(b) \le \|u\|_{[a,b_0]} \quad (2.3)$$

$$u(a) + (b-a)u'(b) \le r\|u\| + r_0, \;\; u'(b) \ge 0 \quad (2.4)$$

$p_i :]a,b[\times]0,+\infty[\to \mathbb{R}_+ \ (i = 0,1,2),\ b_0 \in]a,b[$

$$\|u\|_{[a,b_0]} = \max\left\{u(t) : \ a \le t \le b_0\right\}.$$

$$p_0(t,u(t)) \le -u''(t) \le p_1(t,u(t)) + p_2(t,u(t))(1+u(t)) \quad (2.1)$$

$$u(a) \le r\|u\| + r_0, \quad u(b) \le r\|u\| + r_0 \quad (2.2)$$

Lemma 1. *If*

$$0 < \int_a^b (t-a)(b-t)p_i(t,x)\,dt < +\infty \ \text{ for } x>0 \ (i=0,1), \quad (2.5)$$

$$r<1, \quad \lim_{x\to 0} \int_a^b (t-a)(b-t)p_2(t,x)\,dt < (1-r)(b-a),$$

then there exist a positive constant ρ *and continuous functions* $\varepsilon_i : [a,b] \to [0,+\infty[$ $(i=0,1,2)$ *such that*

$$\begin{gathered} \varepsilon_0(a) = \varepsilon_0(b) = 0, \quad \varepsilon_1(a) = 0, \quad \varepsilon_2(b) = 0, \\ \varepsilon_i(t) > 0 \ \text{ for } \ a<t<b \ \ (i=0,1,2), \end{gathered} \quad (2.6)$$

and an arbitrary solution u *of problem* (2.1), (2.2) *admits the estimates*

$$\varepsilon_0(t) \le u(t) \le \rho \ \text{ for } \ a \le t \le b, \quad (2.7)$$

$$\big|u(t)-u(a)\big| \le \varepsilon_1(t), \ \ \big|u(t)-u(b)\big| \le \varepsilon_2(t) \text{ for } a \le t \le b. \quad (2.8)$$

$$p_0(t,u(t)) \le -u''(t) \le p_1(t,u(t)) + p_2(t,u(t))(1+u(t)) \quad (2.1)$$

$$u(a) \le r\|u\| + r_0, \quad u(b) \le \|u\|_{[a,b_0]} \quad (2.3)$$

Lemma 2. *Let along with* (2.5) *the condition*

$$r < 1, \quad \lim_{x\to+\infty} \int_a^b (t-a)(b-t)p_2(t,x)\,dt < (1-r)(b-b_0)$$

be fulfilled. Then there exist a positive constant ρ *and continuous functions* $\varepsilon_i : [a,b] \to [0,+\infty[$ $(i = 0,1,2)$, *satisfying conditions* (2.6), *such that an arbitrary solution* u *of problem* (2.1), (2.3) *admits estimates* (2.7) *and* (2.8).

$$p_0(t,u(t)) \le -u''(t) \le p_1(t,u(t)) + p_2(t,u(t))(1+u(t)) \quad (2.1)$$

$$u(a) + (b-a)u'(b) \le r\|u\| + r_0, \quad u'(b) \ge 0 \quad (2.4)$$

Lemma 2. *Let*

$$0 < \int_a^b (t-a)p_i(t,x)\,dt < +\infty \ \ \text{for} \ \ x > 0 \ \ (i=0,1),$$

$$r < 1, \quad \lim_{x\to+\infty} \int_a^b (t-a)p_2(t,x)\,dt < (1-r)(b-a).$$

Then there exist a positive constant ρ *and continuous functions* $\varepsilon_i : [a,b] \to [0,+\infty[$ $(i=0,1)$ *and* $\rho_1 :]a,b] \to]0,+\infty[$ *such that*

$$\varepsilon_i(a) = 0, \quad \varepsilon_i(t) > 0 \ \ \text{for} \ \ a < t \le b \ \ (i=0,1),$$

and an arbitrary solution u *of problem* (2.1), (2.4) *admits the estimates*

$$\varepsilon_0(t) \le u(t) \le \rho, \quad 0 \le u'(t) \le \rho_1(t) \ \ \text{for} \ \ a < t \le b,$$
$$\big|u(t) - u(a)\big| \le \varepsilon_1(t) \ \ \text{for} \ \ a \le t \le b.$$