Generalized Linear Differential Equations in a Banach Space and a Continuous Dependence on a Parameter

Giselle Antunes Monteiro and Milan Tvrdý

Institute of Mathematics, Academy of Sciences of the Czech Republic

Brno. 2014

- $-\infty < a < b < \infty$
- X is a Banach space, $\mathcal{L}(X)$ are linear bounded operators on X,
- $G([a,b],X)$ is the Banach space of regulated functions $f:[a,b]\to X$ with the norm $f \in G([a,b],X) \to \|f\|_\infty = \sup_{t\in[a,b]} |f(t)|_X$,
- $(\Delta^- f(a) = \Delta^+ f(b) = 0)$,
- $BV([a,b],X) = \left\{f : [a,b] \to X;\ \mathrm{var}_a^b f < \infty\right\}$,
- $BV([a,b],X) \subset G([a,b],X)$.

$$x(t) = \widetilde{x} + \int_a^t d[A]x + f(t) - f(a) \quad \text{for } t \in [a,b],$$

where

$$\widetilde{x} \in X,\ A \in BV([a,b],\mathcal{L}(X)),\ f \in G([a,b],X) \text{ and } x \in G([a,b],X).$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\,x + f(t) - f(a), \quad t \in [a,b],$$

includes a.o.

- linear functional differential equations with linear impulses,
- linear dynamical equations on time scales.

Notation

- $\mathcal{G} = \{\delta : [a,b] \to (0,1)\}$ are gauges on $[a,b]$.
- $\mathcal{P}=\{P=(D,\xi),\ D=\{a=\alpha_0<\alpha_1<\ldots<\alpha_m=b\},\ \xi=(\xi_1,\ldots,\xi_m)\in\mathbb{R}^m, \xi_j\in[\alpha_{j-1},\alpha_j]\}$ are tagged divisions of $[a,b]$.
- $P=(D,\xi)\in\mathcal{P}[a,b]$ is δ-fine if $[\alpha_{j-1},\alpha_j]\subset(\xi_j-\delta(\xi_j),\xi_j+\delta(\xi_j))$ for all j.
- For $F:[a,b]\to\mathcal{L}(X)$, $g:[a,b]\to X$, $P=(D,\xi)\in\mathcal{P}[a,b]$ we put

$$S(F,dg,P)=\sum_{j=1}^{m}F(\xi_j)\,[g(\alpha_j)-g(\alpha_{j-1})],\quad S(dF,g,P)=\sum_{j=1}^{m}[F(\alpha_{j-1})-F(\alpha_j)]\,g(\xi_j)$$

Definition

$I=\int_a^b F\ d[g]$ if for every $\varepsilon>0$ there exists a $\delta\in\mathcal{G}[a,b]$ such that

$$\Big\|S(F,dg,P)-I\Big\|_X<\varepsilon \text{ for all } \delta\text{-fine tagged divisions } P \text{ of } [a,b]$$

$J=\int_a^b d[F]\,g$ if for every $\varepsilon>0$ there exists a $\delta\in\mathcal{G}[a,b]$ such that

$$\Big\|S(dF,g,P)-J\Big\|_X<\varepsilon \text{ for all } \delta\text{-fine tagged divisions } P \text{ of } [a,b]$$

Basic properties of KS-integral in a Banach space were established by ŠTEFAN SCHWABIK (See also the monograph by CHAIM HÖNIG working with Dushnik integral.)

- KS-integral is linear and is an additive function of interval
- (RS) $\int_a^b d\,[F]g$ exists $\Longrightarrow \int_a^b d\,[F]g$ exists and (RS) $\int_a^b d\,[F]g = \int_a^b d\,[F]g$.
- $F \in G([a,b],\mathcal{L}(X)),\ g \in BV([a,b],X) \Longrightarrow \int_a^b F\,d\,[g]$ exists.
- $F \in BV([a,b],\mathcal{L}(X)),\ g \in G([a,b],X) \Longrightarrow \int_a^b d\,[F]\,g$ exists.
- $F \in G([a,b],\mathcal{L}(X))$ and g is a finite step function $\Longrightarrow \int_a^b d\,[F]\,g$ exists.
- $\int_a^b d[F]\,g$ exists $\Longrightarrow$

$$\Big\| \int_a^b d\,[F(s)]\,g(s) \Big\|_X \le \int_a^b d\,[\mathrm{var}_a^s F]\,g(s) \le (\mathrm{var}_a^b F)\,\|g\|_\infty .$$

- $F \in G([a,b],\mathcal{L}(X)),\ g \in BV([a,b],X),\ \int_a^b d\,[F]\,g$ exists $\Longrightarrow$

$$\|S(d\,F,g,P)\|_X \le 2\,\|F\|_\infty \|g\|_{BV} \text{ for all } P \in \mathcal{P}[a,b]$$

and

$$\Big\| \int_a^b d\,[F]\,g \Big\|_X \le 2\,\|F\|_\infty\,\|g\|_{BV}.$$

Proposition 1 (Existence Theorem)

$$F \in G([a,b],\mathcal{L}(X)), \quad g \in BV([a,b],X) \implies \int_a^b d[F]\,g \text{ exists},$$

$$F \in BV([a,b],\mathcal{L}(X)), \; g \in G([a,b],X) \implies \int_a^b F\,d[g] \text{ exists}.$$

Proposition 2 (Main Convergence Theorem)

ASSUME:

- $g,\, g_k \in G([a,b],X), \quad F, F_k \in BV([a,b],X)$ for $k \in \mathbb{N}$,
- $g_k \rightrightarrows g, \quad F_k \rightrightarrows F$,
- $\alpha^* := \sup\{\mathrm{var}_a^b\, F_k\,; k \in \mathbb{N}\} < \infty$.

THEN: $$\lim_{k\to\infty} \Big\| \int_a^t d[F_k]\,g_k - \int_a^t d[F]\,g \Big\|_X = 0 \quad \text{for } t \in [a,b].$$

$$\text{(GL)} \qquad x(t) = \widetilde{x} + \int_a^t d[A(s)]\,x(s) + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

Proposition 3 [Schwabik, Mathematica Bohemica 1999 and 2000]

ASSUME:

- $A \in BV([a,b], \mathcal{L}(X))$, $f \in G([a,b], X)$,
- $[I - \Delta^{-} A(t)]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$.

THEN:

- (GL) has a unique solution $x \in G([a,b], X)$ on $[a,b]$ for each $\widetilde{x} \in X$, $f \in G([a,b], X)$,
- $x - f \in BV([a,b], X)$.

Gronwall Lemma

ASSUME: $u, p : [a,b] \to [0,\infty)$ are continuous, $K, L \geq 0$ and

$$u(t) \leq K + L \int_a^t (p\,u)\,ds \quad \text{for } t \in [a,b].$$

THEN: $u(t) \leq K \exp\left(L \int_a^t p\,ds\right)$ for $t \in [a,b]$.

Generalized Gronwall Lemma (Schwabik)

ASSUME:

- $u : [a,b] \to [0,\infty)$ is bounded on $[a,b]$, $K, L \geq 0$,
- $h : [a,b] \to [0,\infty)$ is nondecreasing and left-continuos on $(a,b\,]$,
- $u(t) \leq K + L \int_a^t d[h]\,u \quad$ for $t \in [a,b]$.

THEN: $u(t) \leq K \exp\big(L\,[h(t) - h(a)]\big)$ for $t \in [a,b]$.

Corollary of the Generalized Gronwall Lemma

ASSUME: $A \in BV([a,b], \mathcal{L}(X))$, $f \in G([a,b], X)$, $[I - \Delta^- A(t)]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b\,]$ and

$$c_A = \sup\{\|[I - \Delta^- A(t)]^{-1}\|_{L(X)} : t \in (a,b]\}.$$

THEN: $0 < c_A < \infty$ and $\|x(t)\|_X \leq c_A \left(\|\widetilde{x}\|_X + 2\,\|f\|_\infty\right) \exp(\mathrm{var}_a^t A)$ on $[a,b]$

holds for each solution $x \in G([a,b], X)$ of the equation

$$x(t) = \widetilde{x} + \int_a^t d A\,x + f(t) - f(a), \quad t \in [a,b]\,.$$

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\,x + f_k(t) - f_k(a), \quad t \in [a,b].$$
$$x(t) = \widetilde{x} + \int_a^t d[A]\,x + f(t) - f(a), \quad t \in [a,b].$$

THEOREM 1

ASSUME: $A_k, A \in BV([a,b], \mathcal{L}(X))$, $f_k, f \in G([a,b], X)$, $\widetilde{x}_k, \widetilde{x} \in X$ for $k \in \mathbb{N}$,

- $\big[I - \Delta^- A(t)\big]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\operatorname{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

SKETCH OF PROOF: WE ASSUME:

- $A_k, A \in BV([a,b], \mathcal{L}(X)),\ f_k, f \in G([a,b],X),\ \widetilde{x}_k, \widetilde{x} \in X$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N}$, are left-continuous in $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

PUT $w_k = (x_k - f_k) - (x - f)$.

THEN

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

where

$$\widetilde{w}_k = (\widetilde{x}_k - f_k(a)) - (\widetilde{x} - f(a)) \to 0, \quad h_k(t) = \int_a^t d[A_k - A](x-f) + \Big(\int_a^t d[A_k]\, f_k - \int_a^t d[A]\, f\Big),$$

$$\lim_{k\to\infty} \Big\| \int_a^t d[A_k]\, f_k - \int_a^t d[A]\, f \Big\|_X = 0 \quad \text{for } t \in [a,b] \qquad \text{(by Main Convergence Theorem)},$$

$$\Big\| \int_a^t d[A_k - A](x-f) \Big\|_X \le 2\, \|A_k - A\|_\infty\, \|x-f\|_{BV} \quad \text{on } [a,b] \quad \text{(since } (x-f) \in BV([a,b],X)\text{)},$$

TO SUMMARIZE: $\lim_{k\to\infty} \|h_k\|_\infty = 0, \qquad \lim_{k\to\infty} \widetilde{w}_k = 0.$

SKETCH OF PROOF: WE ASSUME:

- $A_k, A \in BV([a,b], \mathcal{L}(X)),\ f_k, f \in G([a,b], X),\ \widetilde{x}_k, \widetilde{x} \in X$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N}$, are left-continuous in $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\mathrm{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

WE HAVE: $w_k = (x_k - f_k) - (x - f)$,

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

$$\lim_{k\to\infty} \|h_k\|_\infty = 0, \qquad \lim_{k\to\infty} \widetilde{w}_k = 0$$

SKETCH OF PROOF: WE ASSUME:

- $A_k, A \in BV([a,b], \mathcal{L}(X)),\ f_k, f \in G([a,b], X),\ \widetilde{x}_k, \widetilde{x} \in X$ for $k \in \mathbb{N}$,
- $A_k,\ k \in \mathbb{N}$, are left-continuous in $(a,b]$,
- $A_k \rightrightarrows A$ on $[a,b]$, $\alpha^* := \sup\{\operatorname{var}_a^b A_k : k \in \mathbb{N}\} < \infty$,
- $\widetilde{x}_k \to \widetilde{x}$, $f_k \rightrightarrows f$ on $[a,b]$.

WE HAVE: $w_k = (x_k - f_k) - (x - f)$,

$$w_k(t) = \widetilde{w}_k + \int_a^t d[A_k(s)]\, w_k(s) + h_k(t) - h_k(a) \quad \text{for } k \in \mathbb{N} \text{ and } t \in [a,b],$$

$$\lim_{k\to\infty} \|h_k\|_\infty = 0, \quad \lim_{k\to\infty} \widetilde{w}_k = 0$$

and by Corollary of the Generalized Gronwall Inequality

$$\|w_k(t)\|_X \leq \left(\|\widetilde{w}_k\|_X + 2\|h_k\|_\infty\right) \exp\left(\operatorname{var}_a^t A_k\right) \text{ on } [a,b] \implies w_k \rightrightarrows 0 \implies x_k \rightrightarrows x. \quad \square$$

$$x_k' = P_k(t)\,x_k, \qquad x_k(a) = \widetilde{x},$$

$$x' = P(t)\,x, \qquad x(a) = \widetilde{x}.$$

Kurzweil & Vorel [Czechoslovak Mathematical Journal, 1957]

ASSUME:

- $\|P_k\|_1 \le p^* < \infty$ for $k \in \mathbb{N}$,
- $\displaystyle\int_a^t P_k\, ds \rightrightarrows \int_a^t P\, ds.$

THEN: $x_k \rightrightarrows x$ on $[a, b]$.

Opial [Journal of Differential Equations, 1967]

ASSUME:

- $\displaystyle\Big(1 + \|P_k\|_1\Big) \Big\| \int_a^t P_k\, ds - \int_a^t P\, ds \Big\|_\infty \rightrightarrows 0.$

THEN: $x_k \rightrightarrows x$ on $[a, b]$.

$$(\mathrm{Eq}_k) \quad x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\, x_k(s), \quad t \in [a,b],$$

$$(\mathrm{Eq}) \quad x(t) = \widetilde{x} + \int_a^t d[A]\, x(s), \quad t \in [a,b].$$

Proposition

<u>ASSUME</u>: $A_k \in BV([a,b], \mathcal{L}(X)), \widetilde{x}_k \in X$ for $k \in \mathbb{N}$,

- $A \in BV([a,b], \mathcal{L}(X)), \widetilde{x} \in X$,
- $[I - \Delta^- A(t)]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $\lim_{k\to\infty} (1 + \mathrm{var}_a^b A_k)\|A_k - A\|_\infty = 0$.

<u>THEN</u>:

- (Eq) has a unique solution $x \in BV([a,b], X)$,
- (Eq_k) has a unique solution $x_k \in BV([a,b], X)$ for $k \in \mathbb{N}$ large enough,
- $x_k \rightrightarrows x$.

Lemma (Kiguradze)

ASSUME:

- $A, A_k \in BV([a,b], \mathcal{L}(X))$ for $k \in \mathbb{N}$, ,
- $[I - \Delta^- A(t)]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $\lim\limits_{k\to\infty} \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty = 0$.

THEN: there are $r^* > 0$ a $k_0 \in \mathbb{N}$ such that

$$\|y\|_\infty \le r^* \left(\|y(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| y(t) - y(a) - \int_a^t d[A_k]\, y \right\|_X \right)$$

for all $y \in G([a,b], X)$ and $k \ge k_0$.

SKETCH OF PROOF: ASSUME: for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\|y_n\|_\infty > n \left(\|y_n(a)\|_X + \left(1 + \operatorname{var}_a^b A_{k_n}\right) \sup_{t \in [a,b]} \left\| y_n(t) - y_n(a) - \int_a^t d\,[A_{k_n}]\, y_n \right\|_X \right).$$

Assume that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\frac{1}{n} > \frac{\|y_n(a)\|_X}{\|y_n\|_\infty} + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t \in [a,b]} \left\| \frac{y_n(t)}{\|y_n\|_\infty} - \frac{y_n(a)}{\|y_n\|_\infty} - \int_a^t d\,[A_{k_n}]\, \frac{y_n}{\|y_n\|_\infty} \right\|_X$$

Assume that for each $n\in\mathbb{N}$ there are $k_n\in\mathbb{N}$ and $y_n\in G([a,b],X)$ such that

$$\left.\begin{array}{l} \frac{1}{n} > \|u_n(a)\|_X + \big(1+\operatorname{var}_a^b A_k\big)\, \sup_{t\in[a,b]} \Big\|u_n(t)-u_n(a)-\displaystyle\int_a^t d\,[A_{k_n}]\,u_n\Big\|_X \\ \text{where}\quad u_n(t) = \frac{y_n(t)}{\|y_n\|_\infty} \quad \text{for } \ t\in[a,b] \ \text{ and } \ n\in\mathbb{N}\,. \end{array}\right\} \implies \|u_n(a)\|_X \to 0\,.$$

Put $\quad v_n(t) = u_n(t) - u_n(a) - \displaystyle\int_a^t d\,[A_{k_n}]\,u_n.\quad$ Then

$$\|v_n\|_\infty < \frac{1}{n\,(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \quad \text{for } n\in\mathbb{N} \implies v_n \rightrightarrows 0\,;$$

$z_n := u_n - v_n \in BV,\ \ z_n(a) = u_n(a),\ \ \|z_n\|_{BV} \le 1+\operatorname{var}_a^b A_{k_n}$ and

$$z_n(t) = z_n(a) + \int_a^t d\,[A]\,z_n + h_n(t),\quad h_n(t) = \int_a^t d\,[A_{k_n}-A]\,z_n + \int_a^t d\,[A_{k_n}]\,v_n \ \text{ for } \ t\in[a,b]\,;$$

$$\left.\begin{array}{l} \Big\|\displaystyle\int_a^t d\,[A_{k_n}-A]\,z_n\Big\|_X \le 2\,\|A_{k_n}-A\|_\infty\,\|z_n\|_{BV} \le 2\,\|A_{k_n}-A\|_\infty\,\big(1+\operatorname{var}_a^b A_{k_n}\big), \\ \Big\|\displaystyle\int_a^t d\,A_{k_n} v_n\Big\|_\infty \le \big(\operatorname{var}_a^b A_{k_n}\big)\,\|v_n\|_X \le \dfrac{1}{n}\,\dfrac{\operatorname{var}_a^b A_{k_n}}{(1+\operatorname{var}_a^b A_{k_n})} \le \dfrac{1}{n} \end{array}\right\} \implies \|h_n\|_\infty \to 0\,.$$

Hence, by the generalized Gronwall inequality

$$\lim_{n\to\infty} \|z_n\|_\infty \le \lim_{n\to\infty} c_A\,\big(\|z_n(a)\|_X + 2\,\|h_n\|_\infty\big) \le \exp\big(c_A\ \operatorname{var}_a^b A\big) = 0\,.$$

Assume that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\left.\begin{array}{l} \frac{1}{n} > \|u_n(a)\|_X + \left(1+\operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| u_n(t) - u_n(a) - \int_a^t d\,[A_{k_n}]\, u_n \right\|_X \\ \text{where} \quad u_n(t) = \frac{y_n(t)}{\|y_n\|_\infty} \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N}. \end{array}\right\} \Longrightarrow \|u_n(a)\|_X \to 0.$$

Put $\quad v_n(t) = u_n(t) - u_n(a) - \int_a^t d\,[A_n]\, u_n.$ Then

$$\|v_n\|_\infty < \frac{1}{n\,(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \quad \text{for } n \in \mathbb{N} \Longrightarrow v_n \rightrightarrows 0;$$

$z_n := u_n - v_n \rightrightarrows 0$

Assume that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\left.\begin{aligned} &\tfrac{1}{n} > \|u_n(a)\|_X + \big(1+\operatorname{var}_a^b A_k\big) \sup_{t\in[a,b]} \Big\| u_n(t) - u_n(a) - \int_a^t d[A_{k_n}]\, u_n \Big\|_X \\ &\text{where} \quad u_n(t) = \tfrac{y_n(t)}{\|y_n\|_\infty} \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N}\,. \end{aligned}\right\} \implies \|u_n(a)\|_X \to 0\,.$$

Put $v_n(t) = u_n(t) - u_n(a) - \int_a^t d[A_n]\, u_n.$ Then

$$\|v_n\|_\infty < \frac{1}{n\,(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \quad \text{for } n \in \mathbb{N} \implies v_n \rightrightarrows 0\,;$$

$z_n := u_n - v_n \rightrightarrows 0 \implies u_n \rightrightarrows 0.$

BUT: $\|u_n\|_\infty = 1$ for all $n \in \mathbb{N}$ - CONTRADICTION!!!

Assume that for each $n \in \mathbb{N}$ there are $k_n \in \mathbb{N}$ and $y_n \in G([a,b],X)$ such that

$$\left.\begin{array}{l} \frac{1}{n} > \|u_n(a)\|_X + \big(1+\operatorname{var}_a^b A_k\big) \sup_{t\in[a,b]} \left\| u_n(t) - u_n(a) - \int_a^t d[A_{k_n}]\, u_n \right\|_X \\ \text{where } \quad u_n(t) = \frac{y_n(t)}{\|y_n\|_\infty} \quad \text{for } t \in [a,b] \text{ and } n \in \mathbb{N}. \end{array}\right\} \implies \|u_n(a)\|_X \to 0.$$

Put $v_n(t) = u_n(t) - u_n(a) - \int_a^t d[A_n]\, u_n$. Then

$$\|v_n\|_\infty < \frac{1}{n\,(1+\operatorname{var}_a^b A_{k_n})} \le \frac{1}{n} \quad \text{for } n \in \mathbb{N} \implies v_n \rightrightarrows 0;$$

$z_n := u_n - v_n \rightrightarrows 0 \implies u_n \rightrightarrows 0.$

BUT: $\|u_n\|_\infty = 1$ for all $n \in \mathbb{N}$ - CONTRADICTION!!! □

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b],\ k \in \mathbb{N}\,,$$

$$x(t) = \widetilde{x} + \int_a^t d[A]\, x(s) + f(t) - f(a), \qquad t \in [a,b]\,,$$

Theorem 3

<u>ASSUME</u>: $A_k, A \in BV([a,b], \mathcal{L}(X))$, $f \in BV([a,b], X)$, $f_k \in G([a,b], X)$, $\widetilde{x}_k, \widetilde{x} \in X$ for $k \in \mathbb{N}$.

- $[I - \Delta^- A(t)]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $\widetilde{x}_k \to \widetilde{x}$,
- $\lim\limits_{k\to\infty} \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty = 0$,
- $\lim\limits_{k\to\infty} \left(1 + \operatorname{var}_a^b A_k\right) \|f_k - f\|_\infty = 0$.

<u>THEN</u>: $x_k \rightrightarrows x$ on $[a,b]$.

$$x_k(t) = \widetilde{x}_k + \int_a^t d[A_k]\, x_k(s) + f_k(t) - f_k(a), \quad t \in [a,b],\ k \in \mathbb{N}\,,$$
$$x(t) = \widetilde{x} + \int_a^t d[A]\, x(s) + f(t) - f(a), \quad t \in [a,b]\,,$$

Theorem 3

ASSUME: $A_k, A \in BV([a,b], \mathcal{L}(X))$, $f \in BV([a,b], X)$, $f_k \in G([a,b], X)$, $\widetilde{x}_k, \widetilde{x} \in X$ for $k \in \mathbb{N}$.

- $[I - \Delta^- A(t)]^{-1} \in \mathcal{L}(X)$ for $t \in (a,b]$,
- $\widetilde{x}_k \to \widetilde{x}$,
- $\lim_{k\to\infty} \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty = 0$,
- $\lim_{k\to\infty} \left(1 + \operatorname{var}_a^b A_k\right) \|f_k - f\|_\infty = 0$.

THEN: $x_k \rightrightarrows x$ on $[a,b]$.

Proof relies on

Lemma (Kiguradze)

$$\|y\|_\infty \le r^* \left(\|y(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| y(t) - y(a) - \int_a^t d[A_k]\, y \right\|_X \right)$$

for all $y \in G([a,b], X)$ and $k \in \mathbb{N}$ large enough.

SKETCH OF PROOF:

$$\|A_k - A\|_\infty \leq \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

SKETCH OF PROOF:

$$\|A_k - A\|_\infty \leq \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

$$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X) \ \text{ on } (a,b] \ \text{ for } k \geq k_1$$

SKETCH OF PROOF:

$$\|A_k - A\|_\infty \le \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

$$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X) \text{ on } (a,b] \text{ for } k \ge k_1$$

$\implies$ (by Schwabik's Proposition 2) $\implies$ solutions x_k, x exist for all $k \ge k_0$.

SKETCH OF PROOF:

$\|A_k - A\|_\infty \leq \left(1+\operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$

$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X)$ on $(a,b]$ for $k \geq k_1$

$\implies$ (by Schwabik's Proposition 2) $\implies$ solutions x_k, x exist for all $k \geq k_0$.

Put $u_k = x_k - x$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - u_k(a) - \int_a^t d[A_k]\, u_k = \int_a^t d\,[A_k - A]\, x + (f_k(t) - f(t)) - (f_k(a) - f(a))\,.$$

SKETCH OF PROOF:

$$\|A_k - A\|_\infty \le \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

$$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X) \text{ on } (a,b] \text{ for } k \ge k_1$$

$\implies$ (by Schwabik's Proposition 2) $\implies$ solutions x_k, x exist for all $k \ge k_0$.

Put $u_k = x_k - x$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - u_k(a) - \int_a^t d[A_k]\, u_k = \int_a^t d\,[A_k - A]\, x + (f_k(t) - f(t)) - (f_k(a) - f(a))\,.$$

Kiguradze's Lemma $\implies \exists\, k_0 \ge k_1$ and $r^* \in (0,\infty)$:

$$\|u_k\|_\infty \le r^* \left(\|u_k(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t\in[a,b]} \left\| u_k(t) - u_k(a) - \int_a^t d\,[A_k]\, u_k \right\|_X \right)$$

$$\le r^* \left(\|\widetilde{x}_k - \widetilde{x}\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \left(2\, \|A_k - A\|_\infty\, \|x\|_{BV} + 2\, \|f_k - f\|_\infty\right)\right) \quad \text{for } k \ge k_0.$$

SKETCH OF PROOF:

$$\|A_k - A\|_\infty \le \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$$

$$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X) \text{ on } (a,b] \text{ for } k \ge k_1$$

$\implies$ (by Schwabik's Proposition 2) $\implies$ solutions x_k, x exist for all $k \ge k_0$.

Put $u_k = x_k - x$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - u_k(a) - \int_a^t d[A_k]\, u_k = \int_a^t d\,[A_k - A]\, x + (f_k(t) - f(t)) - (f_k(a) - f(a))\,.$$

Kiguradze's Lemma $\implies \exists\, k_0 \ge k_1$ and $r^* \in (0,\infty)$:

$$\|u_k\|_\infty \le r^* \left(\|u_k(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t \in [a,b]} \left\| u_k(t) - u_k(a) - \int_a^t d\,[A_k]\, u_k \right\|_X \right)$$

$$\le r^* \left(\|\widetilde{x}_k - \widetilde{x}\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \left(2\, \|A_k - A\|_\infty\, \|x\|_{BV} + 2\, \|f_k - f\|_\infty \right) \right) \quad \text{for } k \ge k_0.$$

$$\implies u_k \rightrightarrows 0$$

SKETCH OF PROOF:

$\|A_k - A\|_\infty \le \left(1 + \operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$

$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X)$ on $(a,b]$ for $k \ge k_1$

$\implies$ (by Schwabik's Proposition 2) $\implies$ solutions x_k, x exist for all $k \ge k_0$.

Put $u_k = x_k - x$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - u_k(a) - \int_a^t d[A_k]\, u_k = \int_a^t d\,[A_k - A]\, x + (f_k(t) - f(t)) - (f_k(a) - f(a)).$$

Kiguradze's Lemma $\implies \exists\, k_0 \ge k_1$ and $r^* \in (0, \infty)$:

$$\|u_k\|_\infty \le r^* \left(\|u_k(a)\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \sup_{t \in [a,b]} \left\| u_k(t) - u_k(a) - \int_a^t d\,[A_k]\, u_k \right\|_X \right)$$

$$\le r^* \left(\|\widetilde{x}_k - \widetilde{x}\|_X + \left(1 + \operatorname{var}_a^b A_k\right) \left(2\, \|A_k - A\|_\infty\, \|x\|_{BV} + 2\, \|f_k - f\|_\infty\right) \right) \quad \text{for } k \ge k_0.$$

$\implies u_k \rightrightarrows 0 \implies x_k \rightrightarrows x$.

SKETCH OF PROOF:

$\|A_k - A\|_\infty \le \left(1 + \mathrm{var}_a^b\, A_k\right) \|A_k - A\|_\infty \to 0 \implies A_k \rightrightarrows A$

$\implies \exists\, k_1 \in \mathbb{N} : [I - \Delta^- A_k(t)]^{-1} \in \mathcal{L}(X)$ on $(a,b]$ for $k \ge k_1$

$\implies$ (by Schwabik's Proposition 2) $\implies$ solutions x_k, x exist for all $k \ge k_0$.

Put $u_k = x_k - x$. Then $u_k(a) = \widetilde{x}_k - \widetilde{x}$ and

$$u_k(t) - u_k(a) - \int_a^t d[A_k]\, u_k = \int_a^t d\,[A_k - A]\, x + (f_k(t) - f(t)) - (f_k(a) - f(a))\,.$$

Kiguradze's Lemma $\implies \exists\, k_0 \ge k_1$ and $r^* \in (0,\infty)$:

$$\|u_k\|_\infty \le r^* \left(\|u_k(a)\|_X + \left(1 + \mathrm{var}_a^b\, A_k\right) \sup_{t\,\in\,[a,b]} \left\| u_k(t) - u_k(a) - \int_a^t d\,[A_k]\, u_k \right\|_X \right)$$

$$\le r^* \left(\|\widetilde{x}_k - \widetilde{x}\|_X + \left(1 + \mathrm{var}_a^b\, A_k\right) \left(2\, \|A_k - A\|_\infty\, \|x\|_{BV} + 2\, \|f_k - f\|_\infty \right)\right) \quad \text{for } k \ge k_0.$$

$\implies u_k \rightrightarrows 0 \implies x_k \rightrightarrows x$. □

Main Theorem could be extended to the case $f \in G([a,b],X)$ if the following convergence assertion was true:

Let $A, A_k \in BV([a,b],\mathcal{L}(X))$ for $k \in \mathbb{N}$ and $\lim_{k\to\infty} \left(1+\operatorname{var}_a^b A_k\right) \|A_k - A\|_\infty = 0.$ Then

$$\int_a^t d[A_k]\,f \rightrightarrows \int_a^t d[A]\,f = 0 \quad \text{for each } f \in G([a,b],X)\,.$$

Let $a=0$, $b=1$, $X=\mathbb{R}$,

$$n_k = [k^{3/2}]+1, \qquad \tau_{m,k} = \frac{1}{2^{n_k-m}} \qquad \text{if } m\in\{0,1,\dots,n_k\},$$

$$a_{0,k} = \frac{2^{n_k}}{k}(-1)^{n_k}, \qquad b_{0,k} = \frac{1}{k}(-1)^{n_k-1},$$

$$a_{m,k} = \frac{2^{n_k-m+1}}{k}(-1)^{n_k-m}, \; b_{m,k} = \frac{3}{k}(-1)^{n_k-m+1} \quad \text{if } m\in\{1,2,\dots,n_k-1\}$$

$$A_k(t) = \begin{cases} 0 & \text{if } t\in[0,\tau_{0,k}], \\ a_{m,k}\,t+b_{m,k} & \text{if } t\in[\tau_{m,k},\tau_{m+1,k}] \text{ and } m\in\{0,1,\dots,n_k-1\}, \end{cases}$$

$$A(t) = 0 \quad \text{for } t\in[0,1].$$

Then

$$\operatorname{var}_0^1 A_k \le \frac{1}{k} + \frac{2\,(n_k-1)}{k} \le \frac{1}{k} + 2\sqrt{k} < \infty,$$

$$\Big(1+\operatorname{var}_0^1 A_k\Big)\,\|A_k-A\|_\infty \le \left(1+\frac{2\,n_k-1}{k}\right)\frac{1}{k} \le \frac{1}{k}+\frac{2}{\sqrt{k}}+\frac{1}{k^2}$$

However, if

$$f(t) = \begin{cases} \dfrac{(-1)^n}{\sqrt[4]{n}} & \text{if } t\in(2^{-n},2^{-(n-1)}] \text{ for some } n\in\mathbb{N}, \\ 0 & \text{if } t=0, \end{cases} \tag{1}$$

then f is regulated, $\operatorname{var}_0^1 f=\infty$ and

$$\int_0^1 d[A_k]\,f \ge \frac{2}{k}\sum_{m=1}^{n_k-1}\frac{1}{\sqrt[4]{m}} > \frac{2}{k}\int_1^{n_k}\frac{1}{\sqrt[4]{t}}\,dt = \frac{8}{3\,k}\Big(\sqrt[4]{(n_k)^3}-1\Big), \tag{2}$$

where the right hand side evidently tends to ∞ for $k\to\infty$.

$\mathbb{T} \subset \mathbb{R}$ is nonempty and closed, $[a,b]_{\mathbb{T}} = [a,b] \cap \mathbb{T}$, $\widetilde{\sigma}(t) := \inf\big([t,b] \cap \mathbb{T}\big)$.

Consider equation

$$(LD) \quad y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,,$$

where $P: [a,b]_{\mathbb{T}} \to \mathbb{R}^{n\times n}$ and $h: [a,b]_{\mathbb{T}} \to X^n$ are rd-continuous in $[a,b]_{\mathbb{T}}$. Put

$$A(t) = \int_a^t P(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{and} \quad f(t) = \int_a^t h(\widetilde{\sigma}(s))\,d[\widetilde{\sigma}(s)] \quad \text{for } t \in [a,b]\,.$$

Lemma (Slavík)

- If $y: [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ is a solution to (LD), then $x = y \circ \widetilde{\sigma}$ is a solution to

$$(\mathrm{GL}) \quad x(t) = \widetilde{y} + \int_a^t d[A]\,x + f(t) - f(a)\,, \quad t \in [a,b]\,.$$

- If x is a solution to (GL) and $y = x|_{\mathbb{T}}$, then y is a solution to (LD).

$$(LD) \quad y(t) = \widetilde{y} + \int_a^t \big[P(s)\,y(s) + h(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,,$$

$$(\mathrm{LD}_k) \quad y(t) = \widetilde{y}_k + \int_a^t \big[P_k(s)\,y(s) + h_k(s)\big]\,\Delta s\,, \quad t \in [a,b]_{\mathbb{T}}\,.$$

Theorem

ASSUME: $P,\ P_k\colon [a,b]_{\mathbb{T}} \to X^{n\times n},\ h,\ h_k\colon [a,b]_{\mathbb{T}} \to \mathbb{R}^n$ for $k \in \mathbb{N}$ are rd-continuous in $[a,b]_{\mathbb{T}}$,

$$\lim_{k\to\infty} \|\widetilde{y}_k - \widetilde{y}\|_{\mathbb{R}^n} = 0\,,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (P_k(s) - P(s))\,\Delta s \Big\|_{\mathbb{R}^{n\times n}} \big[1 + \alpha_k\big] = 0\,,$$

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \Big\| \int_a^t (h_k(s) - h(s))\,\Delta s \Big\|_{\mathbb{R}^{n\times n}} \big[1 + \alpha_k\big] = 0\,,$$

where

$$\alpha_k = \sup_{t\in[a,b]_{\mathbb{T}}} \|P_k(t)\|_{\mathbb{R}^{n\times n}},\ k \in \mathbb{N}\,.$$

THEN: (LD) has a solution y, (LD_k) has a solution y_k for each $k \in \mathbb{N}$ and

$$\lim_{k\to\infty} \sup_{t\in[a,b]_{\mathbb{T}}} \|y_k(t) - y(t)\|_{\mathbb{R}^n} = 0\,.$$

- G. Monteiro and M. Tvrdý. On Kurzweil-Stieltjes integral in Banach space. *Math. Bohem.* **137** (2012), 365–381.
- G. Monteiro and M. Tvrdý. Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. *Discrete Contin. Dyn. Syst.* **33** (1) (2013), 283–303, doi: 10.3934/dcds.2013.33.283.
- Z. Halas, G. Monteiro and M. Tvrdý. Emphatic convergence and sequential solutions of generalized linear differential equations *Mem. Differential Equations Math. Phys.* **54** (2011), 27–49.
- As Ashordia M.. On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Proc. Georgian Acad. Sci. Math.* **1** (1993), 385–394.
- Hakl R., Lomtatidze A. and Stavrolaukis I.P.. On a boundary value problem for scalar linear functional differential equations. *Abstr. Appl. Anal.* **2004** (1) (2004), 45–67, doi:10.1155/S1085337504309061.
- I.T. Kiguradze: *Boundary Value Problems for Systems of Ordinary Differential Equations* (in Russian). Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh. 30 (1987), 3-103; English transl.: J. Sov. Math. **43** (1988), 2259-2339.

- G. Monteiro and M. Tvrdý. On Kurzweil-Stieltjes integral in Banach space. *Math. Bohem.* **137** (2012), 365–381.
- G. Monteiro and M. Tvrdý. Generalized linear differential equations in a Banach space: Continuous dependence on a parameter. *Discrete Contin. Dyn. Syst.* **33** (1) (2013), 283–303, doi: 10.3934/dcds.2013.33.283.
- Z. Halas, G. Monteiro and M. Tvrdý. Emphatic convergence and sequential solutions of generalized linear differential equations *Mem. Differential Equations Math. Phys.* **54** (2011), 27–49.
- As Ashordia M.. On the correctness of linear boundary value problems for systems of generalized ordinary differential equations. *Proc. Georgian Acad. Sci. Math.* **1** (1993), 385–394.
- Hakl R., Lomtatidze A. and Stavrolaukis I.P.. On a boundary value problem for scalar linear functional differential equations. *Abstr. Appl. Anal.* **2004** (1) (2004), 45–67, doi:10.1155/S1085337504309061.
- I.T. Kiguradze: *Boundary Value Problems for Systems of Ordinary Differential Equations* (in Russian). Itogi Nauki Tekh., Ser. Sovrem. Probl. Mat., Novejshie Dostizh. 30 (1987), 3-103; English transl.: J. Sov. Math. **43** (1988), 2259-2339.
- Š. Schwabik. *Generalized Ordinary Differential Equations*. World Scientific, 1992.
- Š. Schwabik. Abstract Perron-Stieltjes integral. *Math. Bohem.* **121**(1996), 425–447.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces. *Math. Bohem.* **124** (1999), 433–457.
- Š. Schwabik. Linear Stieltjes integral equations in Banach spaces II; Operator valued solutions. *Math. Bohem.* **125** (2000), 431–454.