Nonlinear differential systems and regularly varying functions

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Structure of the talk

- Theory of regularly varying functions
- Asymptotic properties of nonlinear differential systems

Theory of regularly varying functions

- initiated by J. Karamata (1930). But there are also earlier works ...
- study of relations such that

$$\lim_{t\to\infty} \frac{f(\lambda t)}{f(t)} = g(\lambda) \in (0,\infty), \quad \forall \lambda > 0,$$

together with their applications (integral transforms – Tauberian theorems, probability theory, analytic number theory, complex analysis, differential equations, etc.)

- N.H. Bingham, C.M. Goldie, J.L. Teugels, *Regular Variation*, Encyclopedia of Mathematics and its Applications, Vol. 27, Cambridge University Press, 1987.
- J. L. Geluk, L. de Haan, *Regular Variation, Extensions and Tauberian Theorems*, CWI Tract 40, Amsterdam, 1987.
- E. Seneta, *Regularly Varying Functions*, Lecture Notes in Mathematics 508, Springer-Verlag, Berlin-Heidelberg-New York, 1976.
- ...

Definition

A measurable function $f : [a, \infty) \to (0, \infty)$ is called regularly varying (at ∞) of index ϑ if

$$\lim_{t \to \infty} \frac{f(\lambda t)}{f(t)} = \lambda^{\vartheta} \text{ for all } \lambda > 0.$$

[Notation: $f \in \mathcal{RV}(\vartheta)$]
If $\vartheta = 0$, then f is called slowly varying.

[Notation: $f \in \mathcal{SV}$]

[$\mathcal{RV}_0$ means regular variation at zero.]

The Uniform Convergence Theorem

If $L \in \mathcal{SV}$, then the relation

$$\lim_{t \to \infty} \frac{L(\lambda t)}{L(t)} = 1$$

holds uniformly on each compact λ-set in $(0, \infty)$.

Representation Theorem

- *f is regularly varying of index ϑ if and only if*

$$f(t) = \varphi(t) \exp\left\{\int_a^t \frac{\delta(s)}{s}\, ds\right\}$$

where $\varphi(t) \to const > 0$ and $\delta(t) \to \vartheta$ as $t \to \infty$.

- *f is regularly varying of index ϑ if and only if*

$$f(t) = t^{\vartheta}\varphi(t) \exp\left\{\int_a^t \frac{\psi(s)}{s}\, ds\right\}$$

where $\varphi(t) \to const > 0$ and $\psi(t) \to 0$ as $t \to \infty$.

If $\varphi(t) \equiv$ const, then f is said to be normalized regularly varying ($f \in \mathcal{NRV}$).

Examples of (non-)$\mathcal{SV}$ functions

f is regularly varying of index ϑ if and only if

$$f(t) = t^{\vartheta} L(t),$$

where $L \in \mathcal{SV}$.

- $\prod_{i=1}^{n} (\ln_i t)^{\mu_i}$, where $\ln_i t = \ln \ln_{i-1} t$ and $\mu_i \in \mathbb{R}$ is $\mathcal{SV}$ function.
- $2 + \sin(\ln_2 t)$ and $(\ln \Gamma(t))/t$ are $\mathcal{SV}$ functions.
- $\frac{1}{t} \int_a^t \frac{1}{\ln s}\, ds$ is $\mathcal{SV}$ function.
- SV functions may exhibit “infinite oscillation” (i.e., $\liminf_{t\to\infty} L(t) = 0$, $\limsup_{t\to\infty} L(t) = \infty$), for example, $\exp\left\{(\ln t)^{\frac{1}{3}} \cos(\ln t)^{\frac{1}{3}}\right\}$.
- $2 + \sin t$, $2 + \sin(\ln t)$ are NOT $\mathcal{SV}$ functions.
- $\exp t$ is NOT $\mathcal{RV}$ function.

- Extension in a logical and useful manner of the class of functions whose asymptotic behavior is that of a power function, to functions where asymptotic behavior is that of a power function multiplied by a factor which varies "more slowly" than a power function.
- $\mathcal{SV} \subset \mathcal{RV}$, but $\mathcal{SV}$ vs. $\mathcal{RV}(\vartheta)$ with $\vartheta \neq 0$
- Regularly varying functions have a "good behavior" with respect to integration resp. summation.
- ...
- Regularly varying functions naturally occur in differential equations.
- ...

Other selected properties

- If $L_1, \dots, L_n \in \mathcal{SV}$, $n \in \mathbb{N}$, and $R(x_1, \dots, x_n)$ is a rational function with positive coefficients, then $R(L_1, \dots, L_n) \in \mathcal{SV}$. In particular,

$$f_1 f_2 \in \mathcal{RV}(\vartheta_1 + \vartheta_2) \text{ and } f_1^{\gamma} \in \mathcal{RV}(\gamma\vartheta_1)$$

  for $f_i \in \mathcal{RV}(\vartheta_i)$, $i = 1, 2$, and $\gamma \in \mathbb{R}$. Moreover, $L_1 \circ L_2 \in \mathcal{SV}$ provided $L_2(t) \to \infty$ as $t \to \infty$.
- If $L \in \mathcal{SV}$ and $\vartheta > 0$, then $t^{\vartheta} L(t) \to \infty$, $t^{-\vartheta} L(t) \to 0$ as $t \to \infty$.
- If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta \le 0$ and $f(t) = \int_t^{\infty} g(s)\, ds$ with g nonincreasing, then

$$\frac{-tf'(t)}{f(t)} = \frac{tg(t)}{f(t)} \to -\vartheta \text{ as } t \to \infty.$$

- If $f \in \mathcal{RV}(\vartheta)$ with $\vartheta \ge 0$ and $f(t) = f(t_0) + \int_{t_0}^{t} g(s)\, ds$ with g monotone, then

$$\frac{tf'(t)}{f(t)} = \frac{tg(t)}{f(t)} \to \vartheta \quad \text{as } t \to \infty.$$

Other selected properties

- *(Almost monotonicity)* For a positive measurable function L it holds: $L \in \mathcal{SV}$ if and only if, for every $\vartheta > 0$, there exist a (regularly varying) nondecreasing function F and a (regularly varying) nonincreasing function G with
$$t^{\vartheta} L(t) \sim F(t) \quad t^{-\vartheta} L(t) \sim G(t) \qquad \text{as } t \to \infty.$$
- *(Asymptotic inversion)* If $g \in \mathcal{RV}(\vartheta)$ with $\vartheta > 0$, then there exists $g \in \mathcal{RV}(1/\vartheta)$ with
$$f(g(t)) \sim g(f(t)) \sim t \quad \text{as } t \to \infty.$$
Here g (an "asymptotic inverse" of f) is determined uniquely to within asymptotic equivalence. One version of g is the generalized inverse $f^{\leftarrow}(t) := \inf\{s \in [a, \infty) : f(s) > t\}$.

Other selected properties (Karamata's theorem!!)

- *(Karamata's theorem; direct half)* If $L \in \mathcal{SV}$, then

$$\int_t^\infty s^\zeta L(s)\, ds \sim \frac{1}{-\zeta - 1} t^{\zeta+1} L(t)$$

provided $\zeta < -1$, and

$$\int_a^t s^\zeta L(s)\, ds \sim \frac{1}{\zeta + 1} t^{\zeta+1} L(t)$$

provided $\zeta > -1$. The integral $\int_a^\infty L(s)/s\, ds$ may or may not converge. The function $\tilde{L}(t) = \int_a^t L(s)/s\, ds$ is a new SV function and such that $L(t)/\tilde{L}(t) \to 0$ as $t \to \infty$.

- *(Karamata's theorem; converse half)* If for some $\sigma < -(\zeta + 1)$,

$$t^{\sigma+1} f(t) \Big/ \int_t^\infty s^\sigma f(s)\, ds \to -(\sigma + \zeta + 1) \text{ as } t \to \infty, \text{ then } f \in \mathcal{RV}(\zeta).$$

If for some $\sigma > -(\zeta + 1)$,

$$t^{\sigma+1} f(t) \Big/ \int_a^t s^\sigma f(s)\, ds \to \sigma + \zeta + 1 \text{ as } t \to \infty, \text{ then } f \in \mathcal{RV}(\zeta).$$

Many related topics

- Regular variation at zero, rapid variation, regular boundedness, de Hann class, Zygmund class, ...
- Discrete variable, time scale variable, complex variable, complex values, higher dimensions, topological groups, ...
- Probability theory, complex analysis, Abelian theorems, Tauberian theorems, Mercerian theorems, analytic number theory, differential equations, difference equations, functional equations, game theory, ...

Genesis of the problem

- Kamo and Usami (2000, 2001) considered the quasilinear (or the generalized Emden-Fowler) equation

$$(\Phi_\alpha(y'))' = p(t)\Phi_\beta(y), \quad \Phi_\lambda(u) = |u|^\lambda \operatorname{sgn} u, \tag{E}$$

where $\alpha, \beta > 0$, $p(t) \sim t^\sigma$. Under additional conditions on α, β, σ, they showed that certain solutions y of (E) have the form

$$y(t) \sim Kt^\gamma, \quad \text{where } K = K(\alpha, \beta, \sigma), \gamma = \gamma(\alpha, \beta, \sigma).$$

The key role was played by the asymptotic equivalence theorem which says, roughly speaking:

If the coefficients of two equations of the form (E) are asymptotically equivalent, then their solutions which are "of the same type" are also asymptotically equivalent.

- We studied regular variation (in connection with the investigation of difference and dynamic equations) and pioneering works on differential equations in the framework of RV by Geluk, Kusano, Marić, Tomić, Omey, ...

We studied asymptotics for nonlinear systems.

Genesis of the problem

Is a generalization possible?

- A general coefficient in the differential term?
- Regularly varying coefficients?
- Regularly varying nonlinearities?
- Coupled systems?
- Second order systems of k equations (even-order scalar equations)?
- First order systems of n equations (n may be even or odd)?

– An asymptotic equivalence theorem cannot be used.
– The theory of RV is widely used.

Nonlinear system (of Emden-Fowler type)

$$\begin{cases} x_1' &= a_1(t)F_1(x_2), \\ x_2' &= a_2(t)F_2(x_3), \\ &\vdots \\ x_{n-1}' &= a_{n-1}(t)F_{n-1}(x_n), \\ x_n' &= a_n(t)F_n(x_1), \end{cases} \tag{S}$$

$n \in \mathbb{N}$, $n \geq 2$.

- a_i are continuous, eventually of one sign, and

$$|a_i| \in \mathcal{RV}(\sigma_i), \quad \sigma_i \in \mathbb{R}, \quad i = 1, \dots, n,$$

- F_i are continuous with $uF_i(u) > 0$ for $u \neq 0$, and

$$|F_i(|\cdot|)| \in \mathcal{RV}(\alpha_i), \text{ resp. } |F_i(|\cdot|)| \in \mathcal{RV}_0(\alpha_i), \; \alpha_i \in (0, \infty), \; i = 1, \dots, n,$$

$\mathcal{RV}_0$ being regular variation at zero.

Special cases – *n*-th order two term nonlinear DE's

- (Kiguradze, Chanturia, ...)

$$x^{(n)} = p(t)\Phi_\beta(x)$$

(... useful also for comparison purposes ...)

- (Naito, ...)

$$D(\gamma_n)D(\gamma_{n-1})\cdots D(\gamma_1)x(t) = p(t)\Phi_\beta(x),$$

where $D(\gamma)x(t) = \frac{d}{dt}(\Phi_\gamma(x))$

- More general cases (the order can be even as well as odd):

$$D_{q_n}(\gamma_n)D_{q_{n-1}}(\gamma_{n-1})\cdots D_{q_1}(\gamma_1)x(t) = p(t)\Phi_\beta(x),$$

where $D_{q_i}(\gamma_i)x(t) = \frac{d}{dt}(q_i(t)\Phi_{\gamma_i}(x))$

- $n = 2$ resp. $n = 4$: Frequently studied 2nd order resp. 4th order differential equations.

Special cases – equations with a generalized Laplacian and/or an $\mathcal{RV}$ nonlinearity on the RHS

For example, the second order equation

$$(r(t)G(x'))' = p(t)F(x),$$

with

- a generalized Laplacian (may include the classical *p*-Laplacian operator or the curvature operator or the relativity operator or ...)
- a regularly varying nonlinearity on the right-hand side

Typical examples of nonlinearities (we do not require monotonicity):

- $F_i(u) = \Phi_\alpha(u) = |u|^\alpha \operatorname{sgn} u$ (for second order equations or systems it may lead to classical Laplacian operator).
- $F_i(u) = \Phi_\alpha(u)L(u)$, with $\alpha > 0$, where $L(u) \to c \in (0, \infty)$, or $L(u) = |\ln u|^{\gamma_1} |\ln|\ln u||^{\gamma_2}$.
- $F_i(u) = u^\alpha(A + Bu^\beta)^\gamma$; a special choice yields $u/\sqrt{1+u^2}$ or $u/\sqrt{1-u^2}$.
- Possible inverses or asymptotic inverses of such nonlinearities can also be considered. (The Lambert W function may play a role.)

Partial differential systems

System (S) includes also second order nonlinear systems of the form

$$\begin{cases} (A_1(t)\Phi_{\lambda_1}(y_1'))' = B_1(t)G_1(y_2), \\ (A_2(t)\Phi_{\lambda_2}(y_2'))' = B_2(t)G_2(y_3), \\ \qquad \vdots \\ (A_k(t)\Phi_{\lambda_k}(y_k'))' = B_k(t)G_k(y_1), \end{cases}$$

which play important role in the study of positive radial solutions to the partial differential system

$$\begin{cases} \mathrm{div}(\|\nabla u_1\|^{\lambda_1 - 1}\nabla u_1) = \varphi_1(\|z\|)G_1(u_2), \\ \mathrm{div}(\|\nabla u_2\|^{\lambda_2 - 1}\nabla u_2) = \varphi_2(\|z\|)G_2(u_3), \\ \qquad \vdots \\ \mathrm{div}(\|\nabla u_k\|^{\lambda_k - 1}\nabla u_k) = \varphi_k(\|z\|)G_k(u_1). \end{cases}$$

Systems of Lane-Emden type (Dalmasso, ...).

For $k = 2$ we get (well-studied) coupled systems.

Extreme solutions

- $\mathcal{DS}$ – the set of all (proper) solutions of (S) whose components are eventually positive and decreasing.
- $\mathcal{IS}$ – the set of all (proper) solutions of (S) whose components are eventually positive and increasing.

Relations with some known concepts (Kiguradze type classification, Kneser solutions, fast growing solutions, ...)

The asymptotic behavior (and the existence) of $\mathcal{DS}$ and $\mathcal{IS}$ solutions where at least one of the components tends to a positive constant, is – from a certain point of view – clear. Hence we focus to the following extreme classes:

- $\mathcal{SDS} = \{(x_1, \dots, x_n) \in \mathcal{DS} : \lim_{t\to\infty} x_i(t) = 0,\ i = 1, \dots, n\}$; the so-called strongly decreasing solutions.
- $\mathcal{SIS} = \{(x_1, \dots, x_n) \in \mathcal{IS} : \lim_{t\to\infty} x_i(t) = \infty,\ i = 1, \dots, n\}$; the so-called strongly increasing solutions.

Sign conditions on a_i

Consider the system

$$x_i' = a_i(t)F_i(x_{i+1}), \quad i = 1, \dots, n. \tag{S}$$

x_{n+1} means x_1.

We study

- the set $\mathcal{DS}$ under the condition $\operatorname{sgn} a_i = -1$, $i = 1, \dots, n$.
- the set $\mathcal{IS}$ under the condition $\operatorname{sgn} a_i = 1$, $i = 1, \dots, n$.

This is somehow natural and non-restrictive setting because:

... explanation via a Kiguradze type of classification of solutions and the conditions for the existence in these classes ...

Our sign conditions posed on the coefficients in (S) are exactly those which allow the existence of $\mathcal{DS}$ resp. $\mathcal{IS}$ solutions.

- Our aim (at this stage) is not to make a complete classification and to discuss the existence and behavior in each class.
- Rather we chose two “difficult classes” of solutions to a quite general system and we try to find an efficient METHOD which utilizes the theory of regular variation to describe asymptotic behavior of such solutions.

$\mathcal{SIS}$ solutions of nonlinear systems

$$x_i' = a_i(t)F_i(x_{i+1}), \quad i = 1, \dots, n. \tag{S}$$

- Conditions on coefficients: $a_i > 0$, $a_i \in \mathcal{RV}(\sigma_i)$, $i = 1, \dots, n$.
- Conditions on nonlinearities: $F_i \in \mathcal{RV}(\alpha_i)$, $L_{F_i}(ug(u)) \sim L_{F_i}(u)$ as $u \to \infty$, for every $g \in \mathcal{SV}$, where $L_{F_i}(t) = t^{-\alpha_i}F_i(t)$, $i = 1, \dots, n$. Nonlinearities do not need to be monotone.
- "Subhomogeneity": $\alpha_1 \cdots \alpha_n < 1$
- Convention for subscripts: By a subscript $k \in \mathbb{N}$ we mean $k = i$, where $i \in \{1, \dots, n\}$ and $k \equiv i \pmod{n}$. Then, for a subscript k,

$$k = \begin{cases} k & \text{if } k \le n, \\ k - mn & \text{if } k > n, \end{cases}$$

where $m \in \mathbb{N}$ is such that $1 \le k - mn \le n$.

Notation

- $(\nu_1, \dots, \nu_n)$ is the unique solution of the system

$$\nu_i - \alpha_i \nu_{i+1} = \sigma_{i+1} + 1, \quad i = 1, \dots, n.$$

$$\left(\text{Explicit form: } \nu_i = \frac{1}{1 - \alpha_1 \cdots \alpha_n} \sum_{k=0}^{n-1} \left((\sigma_{i+k} + 1) \prod_{j=i}^{i+k-1} \alpha_j \right), \quad i = 1, \dots, n. \right)$$

- $(L_1(t), \dots, L_n(t)) \in \mathcal{SV}^n$ is the unique solution (up to asymptotic equivalence) of the system

$$L_i(t) \sim L_{a_i}(t) L_{i+1}^{\alpha_i}(t) L_{F_i}(t^{\nu_i+1}) \quad \text{as } t \to \infty, i = 1, \dots, n,$$

where $L_{F_i}(t) = t^{-\alpha_i} F_i(t)$, $L_{a_i}(t) = t^{-\sigma_i} a_i(t)$. (Explicit form can be given.)

- $(h_1, \dots, h_n)$ is the unique solution of the system

$$|\nu_i| h_i = h_{i+1}^{\alpha_i}, \quad i = 1, \dots, n.$$

(Explicit form can be given.)

Theorem

- *If* $\nu_i > 0$, $i = 1, \dots, n$, *then there exists*

$$(x_1, \dots, x_n) \in \mathcal{SIS} \cap (\mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n))$$

and (for every such a solution)

$$x_i(t) \sim K_i t^{\nu_i} L_i(t) \quad \text{as } t \to \infty, \text{ i=1,\dots,n.} \tag{AF}$$

- *If* $\nu_i > 0$ *and , in addition,* $F_i = \Phi_{\alpha_i}$, $i = 1, \dots, n$, *then* $\mathcal{SIS} \neq \emptyset$ *and for EVERY* $(x_1, \dots, x_n) \in \mathcal{SIS}$, *one has*

$$(x_1, \dots, x_n) \in \mathcal{RV}(\nu_1) \times \cdots \times \mathcal{RV}(\nu_n)$$

and (AF) holds with $L_{F_1} \equiv \cdots \equiv L_{F_n} \equiv 1$.

Proof (of the first part)

Properties of $\mathcal{RV}$ functions are frequently used ...

- The Schauder-Tychonoff fixed point theorem: We obtain a solution $(x_1, \dots, x_n) \in \mathcal{SIS}$ such that $c_i t^{\nu_i} L_i(t) \le x_i(t) \le d_i t^{\nu_i} L_i(t)$ for some constants c_i, d_i, $i = 1, \dots, n$.
- $\liminf_{t\to\infty} x_i(\lambda t)/x_i(t), \limsup_{t\to\infty} x_i(\lambda t)/x_i(t) \in (0, \infty)$.
- The uniform convergence theorem and the generalized L'Hospital rule yield that $\lim_{t\to\infty} x_i(\lambda t)/x_i(t)$ exists; in fact,

  $$\limsup_{t\to\infty} x_i(\lambda t)/x_i(t) \le \lambda^{\nu_i} \le \liminf_{t\to\infty} x_i(\lambda t)/x_i(t).$$

  Thus, $\mathcal{RV}$ follows.
- Playing with asymptotic relations and the Karamata theorem yield the asymptotic formula.

Proof (of the second part)

- $\mathcal{SIS} \neq \emptyset$ follows from the previous part. We take an arbitrary $(x_1, \ldots, x_n) \in \mathcal{SIS}$.
- $\nu_i > 0$ for all $i = 1, \ldots, n$ implies that, for some $m \in \{1, \ldots, n\}$, it holds

$$\begin{cases} \varrho_{n-1}^{[m]} := \sigma_{m+n-1} + 1 > 0, \\ \varrho_{n-2}^{[m]} := \sigma_{m+n-2} + 1 + \alpha_{m+n-2}\varrho_{n-1}^{[m]} > 0, \\ \varrho_{n-3}^{[m]} := \sigma_{m+n-3} + 1 + \alpha_{m+n-3}\varrho_{n-2}^{[m]} > 0, \\ \quad \vdots \\ \varrho_{1}^{[m]} := \sigma_{m+1} + 1 + \alpha_{m+1}\varrho_{2}^{[m]} > 0, \\ \varrho_{0}^{[m]} := \sigma_{m} + 1 + \alpha_{m}\varrho_{1}^{[m]} > 0. \end{cases} \tag{FUJ}$$

- The Karamata Theorem and (FUJ) – we work with (S) in a (scalar) integral form – yield $x_i(t) \leq d_i t^{\nu_i} L_i(t)$, $i = 1, \ldots, n$.

Proof (of the second part) – continuation

- The generalized AM-GM inequality

$$\frac{1}{p}\sum_{i=1}^{n} p_i u_i \geq \left(\prod_{i=1}^{n} u_i^{p_i}\right)^{\frac{1}{p}},$$

  $p = p_1 + \cdots + p_n$, the Karamata integration theorem, properties of $\mathcal{RV}$ functions, and the estimates from the previous part yield $x_i(t) \geq c_i t^{\nu_i} L_i(t)$, $i = 1, \ldots, n$.

- $x_i \in \mathcal{RV}(\nu_i)$ and the asymptotic formula follow by the same arguments as in the first part.

$\mathcal{SDS}$ solutions of nonlinear systems

- We assume $F_i \in \mathcal{RV}_0(\alpha_i)$ and relevant conditions on their $\mathcal{SV}_0$ components.
- We assume $\nu_i < 0$.
- Asymptotic formula takes the same form.
- The proof uses similar ideas.
- In the proof of $\mathcal{SDS} \neq \emptyset$ we can even use a general approach (instead of regular variation of the coefficients we assume certain integral conditions). We apply the Schauder-Tychonoff fixed point theorem to a family of auxiliary problems $\to$ we get a sequence of solutions $\to$ we then utilize the fact that a singular (extinct) solution exists; this helps us to show that the solution obtained from the sequence is positive.

Special cases – n-th order equations

Let $p(t) = t^{\varrho} L_p(t)$ with $L_p \in \mathcal{SV}$ and $0 < \beta < 1$.

- ($\mathcal{SDS}$ solutions.) If $\varrho + n < 0$, then the equation $x^{(n)} = (-1)^n p(t)\Phi_\beta(x)$ possesses a solution x such that $\lim_{t\to\infty} x^{(i)}(t) = 0$, $i = 0, \dots, n-1$, and for any such a solution there hold

$$x \in \mathcal{RV}\left(\frac{\varrho + n}{1-\beta}\right) \quad\text{and}\quad x^{1-\beta}(t) \sim t^{\varrho+n} L_p(t) \prod_{j=1}^{n} \frac{1-\beta}{-\varrho - n + (1-\beta)(j-1)}.$$

- ($\mathcal{SIS}$ or fast growing solutions.) If $\varrho + 1 + \beta(n-1) > 0$, then the equation $x^{(n)} = p(t)\Phi_\beta(x)$ possesses a solution x such that $\lim_{t\to\infty} x^{(i)}(t) = \infty$, $i = 0, \dots, n-1$, and for any such a solution there hold

$$x \in \mathcal{RV}\left(\frac{\varrho + n}{1-\beta}\right) \quad\text{and}\quad x^{1-\beta}(t) \sim t^{\varrho+n} L_p(t) \prod_{j=1}^{n} \frac{1-\beta}{\varrho + n - (1-\beta)(j-1)}.$$

The regular variation of all these solutions is normalized.

Special cases – nonlinear 2nd order equations

Consider the equation

$$(r(t)G(x'))' = p(t)F(x), \tag{E}$$

- $r \in \mathcal{RV}(\sigma), p \in \mathcal{RV}(\varrho)$,
- $uF(u) > 0, uG(u) > 0$ for $u \neq 0$, $|G(|\cdot|)| \in \mathcal{RV}_0(\alpha)$, $|F(|\cdot|)| \in \mathcal{RV}_0(\beta)$,
- $G^{\leftarrow}$ denotes a generalized inversion of G,
- $L_F(ug(u)) \sim L_F(u),\ L_{G^{\leftarrow}}(ug(u)) \sim L_{G^{\leftarrow}}(u)$ as $u \to 0+$, for all $g \in \mathcal{SV}_0$.

If

$$\alpha > \beta \ \text{ and } \ \varrho + 1 < \min\left\{\sigma - \alpha, \beta(\sigma - \alpha)/\alpha\right\},$$

then (E) possesses an eventually positive decreasing solution x such that

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} r(t)G(x'(t)) = 0, \quad x \in \mathcal{RV}(\nu), \ \text{(and for any such a solution)}$$

$$x^{\alpha-\beta}(t) \sim \frac{1}{-(\varrho + 1 + \beta\nu)(-\nu)^{\alpha}} \cdot \frac{L^{\alpha}_{G^{\leftarrow}}(t^{\varrho+1+\beta\nu-\sigma})L_F(t^{\nu})L_p(t)}{L_r(t)} \cdot t^{\nu(\alpha-\beta)}$$

as $t \to \infty$, where $\nu = (\alpha - \sigma + \varrho + 1)/(\alpha - \beta)$.

Examples of G:

$$G(u) = \Phi_{\alpha}(u), \ \text{ or } \ G(u) = \frac{u}{\sqrt{1+u^2}}, \ \text{ or } \ G(u) = \frac{u}{\sqrt{1-u^2}}.$$

Possible non-regularly varying coefficients

– new independent variable $s = \zeta(t)$, $\zeta'(t) \neq 0$
– new vector function $(w_1, \dots, w_n)(s) = (x_1, \dots, x_n)(t)$.

$\rightarrow$ (S) is transformed into the system

$$\frac{d}{ds} w_i = \hat{a}_i(s) F_i(w_{i+1}), \quad \text{where} \quad \hat{a}_i = \frac{a_i \circ \zeta^{-1}}{\zeta' \circ \zeta^{-1}}, \quad i = 1, \dots, n. \qquad \text{(TS)}$$

If $\hat{a}_i \in \bigcup_{\vartheta \in \mathbb{R}} \mathcal{RV}(\vartheta)$, $i = 1, \dots, n$, then our results can be applied to (TS), although the original system may have non-$\mathcal{RV}$ coefficients.

For example:

- Set $a_i(t) = e^{\gamma_i t} g_i(t)$, where $\gamma_i \in \mathbb{R}$, $g_i \in \bigcup_{\vartheta \in \mathbb{R}} \mathcal{RV}(\vartheta)$, $i = 1, \dots, n$
- Set $\zeta(t) = e^t$

Then

- $a_i \notin \bigcup_{\vartheta \in \mathbb{R}} \mathcal{RV}(\vartheta)$ provided $\gamma_i \neq 0$
- $\hat{a}_i(s) = s^{\gamma_i - 1}(g_i \circ \ln)(s) \in \mathcal{RV}(\gamma_i - 1)$, $i = 1, \dots, n$.

Related works

- ..., Kiguradze, Chanturia, ...Edelson, ..., Kamo, Usami, ..., Clément, Manásevich, Mitidieri,..., Naito, ...,Dalmasso, ... (classification, existence of solutions to various problems, asymptotics, ...)
- (existence of (nearly) $\mathcal{RV}$ solutions, asymptotic formulas, complete classification of solutions (in the framework of $\mathcal{RV}$),...)
  - Avakumović (1947, Thomas-Fermi equation, probably the first application of RV in DE's.)
  - Evtukhov, Kharkov, Samoilenko, Vladova, ...
  - Jaroš, Kusano, Manojlović, Marić, Tanigawa, ...

Novelties and advantages; comparison

- In addition to existence, we show that EVERY solution (of a given type) is RV and a precise asymptotic formula is established. The method seems to have a potential for working also in some other cases.
- System (S) is quite general and includes various types of nonlinear equations and systems
- In some of the previous works only the existence of solutions (to more special equations) $x(t) \asymp f(t)$, where $f \in \mathcal{RV}(\nu)$, is showed. We have a more precise information: $x \in \mathcal{RV}(\nu)$.
- We do not need to distinguish among particular cases concerning typical integral conditions. All possible eventualities are included in one statement (one proof).
  For example: The equation $(r(t)\Phi_\alpha(x'))' + p(t)\Phi_\beta(x) = 0$ and the condition $\int^\infty r^{-1/\alpha}(s)\,ds = \infty$, resp. $\int^\infty r^{-1/\alpha}(s)\,ds < \infty$.
- Results are new also in the scalar 2nd order case.

Future research directions

Applications of the theory of RV functions

- Nonlinear system (S): Superlinearity, other types of solutions, singular ($\mathcal{RV}$) nonlinearities, ...
- A more precise description of asymptotic behavior (de Haan classes, ...)
- Other types of ordinary differential systems and equations, for example:

  $$(r(t)G(y'))' = p(t)F(y),$$

  F, G are $\mathcal{RV}$ functions with the SAME indices.
  Nearly half-linear or nearly linear equations.
  ("Neither superlinear, nor sublinear.")
- Other types of equations: delay, difference, dynamic, partial, ...
- ...

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