Dimension of the solution set of the homogeneous first-order linear functional differential equation

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- U set of solutions to (1)
- $1 \leq \dim U < +\infty$

Theorem

$$u' = \ell(u)(t) + q(t) \qquad \text{for a. e. } \; t \in [a,b], \qquad h(u) = c$$

is uniquely solvable for every $q \in L\big([a,b];\mathbb{R}\big)$, $c \in \mathbb{R}$ iff

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$h : C\big([a, b]; \mathbb{R}\big) \to \mathbb{R}$ linear bounded

Definition

$\ell \in \mathcal{S}_{ab}(a)$

$$\left.\begin{aligned} & u \in AC\big([a,b];R\big), \\ & u'(t) \geq \ell(u)(t) \quad \text{for} \;\; t \in [a,b], \\ & u(a) \geq 0 \end{aligned}\right\} \implies \quad u(t) \geq 0 \qquad \text{for} \;\; t \in [a,b]$$

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$$\mathcal{S}'_{ab}(a) \subset \mathcal{S}_{ab}(a)$$

Definition

$\ell \in \mathcal{S}_{ab}(b)$

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$$\mathcal{S}'_{ab}(b) \subset \mathcal{S}_{ab}(b)$$

Definition

$\ell \in \mathcal{P}_{ab}^+$

$$\left.\begin{array}{l} u \in AC\big([a,b];\mathbb{R}\big) \\ u(t) \geq 0 \quad \text{for } \ t \in [a,b] \\ u'(t) \geq 0 \quad \text{for a. e. } \ t \in [a,b] \end{array}\right\} \Longrightarrow \quad \ell(u)(t) \geq 0 \quad \text{for a. e. } \ t \in [a,b]$$

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Definition

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Sets $\mathcal{P}_{ab}^+$ and $\mathcal{P}_{ab}^-$

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$$\mathcal{P}_{ab} \subset \mathcal{P}_{ab}^+ \cap \mathcal{P}_{ab}^-, \qquad \mathcal{P}_{ab} \neq \mathcal{P}_{ab}^+ \cap \mathcal{P}_{ab}^-$$

Arguments Deviation

$$\ell(u)(t) = p(t)u(\tau(t)) - g(t)u(\mu(t))$$

- $p, g \in L\big([a, b]; \mathbb{R}_+\big)$
- $\tau, \mu : [a, b] \to [a, b]$ . . . measurable functions

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Proposition

$\ell \in \mathcal{P}^+_{ab}$ iff

$$p(t) \geq g(t), \qquad g(t)\big(\tau(t) - \mu(t)\big) \geq 0, \qquad t \in [a, b]$$

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Theorem 1

Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Then

- $\ell \in \mathcal{S}'_{ab}(a)$,

Theorem 1

Let $\ell \in \mathcal{P}_{ab}^{+}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Then

- $\ell \in \mathcal{S}'_{ab}(a)$,
- $\dim\ U = 1$ and the set U is generated by a positive nondecreasing function.

$$\ell(u)(t) = p(t)u(\tau(t)) - g(t)u(\mu(t))$$

Corollary

Let

$$p(t) \geq g(t) \qquad \text{for a. e. } t \in [a, b],$$
$$g(t)\big(\tau(t) - \mu(t)\big) \geq 0 \qquad \text{for a. e. } t \in [a, b].$$

Let, moreover, either

$$\int_a^b p(t)dt < 1$$

or

$$\int_t^{\tau(t)} p(s)ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a, b].$$

Then the conclusion of Theorem 1 holds.

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Let, moreover, either

$$\int_a^b \left(p(t) - g(t) + g(t) \int_{\mu(t)}^{\tau(t)} p(s)ds \right) dt < 1$$

or τ be a nondecreasing continuous function with $\tau(t) \geq t$,

$$\int_t^{\tau(\tau(t))} \left(p(s) - g(s) + g(s) \int_{\mu(s)}^{\tau(s)} p(\xi)d\xi \right) ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a,b].$$

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Theorem 2

Let $\ell \in \mathcal{P}_{ab}^{+}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, there exist $\gamma \in AC\big([a,b];\mathbb{R}\big)$ satisfying

$$\gamma(t) > 0 \qquad \text{for } \ t \in [a,b],$$
$$\gamma'(t) \geq \ell(\gamma)(t) \qquad \text{for a. e. } \ t \in [a,b].$$

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$$g(t)\big(\mu(t) - t\big) \geq 0 \qquad \text{for a. e. } t \in [a, b].$$

Let, moreover, either

$$\int_a^b p(t) \exp\left(-\int_t^{\tau(t)} g(s)ds\right) dt < 1$$

or

$$\int_t^{\tau(t)} p(s) \exp\left(-\int_s^{\tau(s)} g(\xi)d\xi\right) ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a, b].$$

Then the conclusion of Theorem 2 holds.

Theorem 3

Let $\ell \in \mathcal{P}_{ab}^{-}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Let, moreover, there exist $\gamma \in AC\big([a,b];\mathbb{R}\big)$ satisfying

$$\gamma(t) > 0 \qquad \text{for } \ t \in [a,b],$$
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Theorem 3

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$$\gamma(t) > 0 \qquad \text{for } \ t \in [a,b],$$
$$\gamma'(t) \geq \ell(\gamma)(t) \qquad \text{for a. e. } \ t \in [a,b].$$

Then

- $\ell \in \mathcal{S}_{ab}(a)$,
- $\dim\ U = 1$ and the set U is generated by a positive function u with the following property:

  $$u(a) = \min\big\{u(t) : t \in [a,b]\big\}$$

  and, in addition, if there exists $c \in]a,b]$ such that $u(c) = u(a)$ then

  $$u(t) = u(a) \qquad \text{for } \ t \in [a,c].$$

$$\ell(u)(t) = p(t)u(\tau(t)) - g(t)u(\mu(t))$$

Corollary

Let

$$p(t) \geq g(t) \qquad \text{for a. e. } \ t \in [a, b],$$
$$g(t)\big(\tau(t) - \mu(t)\big) \leq 0 \qquad \text{for a. e. } \ t \in [a, b],$$
$$g(t)\big(\mu(t) - t\big) \leq 0 \qquad \text{for a. e. } \ t \in [a, b].$$

Let, moreover,

$$\int_a^b g(t)dt < 1, \qquad \int_a^b \big(p(t) - g(t)\big)dt < 1$$

Then the conclusion of Theorem 3 holds.

$$\ell(u)(t) = p(t)u(\tau(t)) - g(t)u(\mu(t))$$

Corollary

Let

$$p(t) \geq g(t) \qquad \text{for a. e. } t \in [a,b],$$
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$$g(t)\big(\mu(t) - t\big) \leq 0 \qquad \text{for a. e. } t \in [a,b].$$

Let, moreover,

$$\int_{\mu(t)}^{t} g(s)ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a,b],$$
$$\int_{t}^{\tau(t)} \big(p(s) - g(s)\big)ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a,b].$$

Then the conclusion of Theorem 3 holds.

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$$g(t)\big(\mu(t) - t\big) \leq 0 \qquad \text{for a. e. } t \in [a,b].$$

Let, moreover, τ be a nondecreasing continuous function with $\tau(t) \leq t$,

$$\int_{\tau(\tau(t))}^{t} g(s) \int_{\tau(s)}^{\mu(s)} p(\xi)d\xi ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a,b],$$
$$g(t) \int_{\tau(t)}^{\mu(t)} \big(p(s) - g(s)\big)ds \leq p(t) - g(t) \qquad \text{for a. e. } t \in [a,b].$$

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Theorem 4

Let $\ell \in \mathcal{P}_{ab}^{-}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$ be a b-Volterra operator. Then

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- $\ell \in \mathcal{S}_{ab}(b)$,
- $\dim\ U = 1$ and the set U is generated by a positive function u with the following property:

$$u(b) = \max \left\{ u(t) : t \in [a, b] \right\}$$

and, in addition, if there exists $c \in [a, b[$ such that $u(c) = u(b)$ then

$$u(t) = u(b) \qquad \text{for } \ t \in [c, b].$$

$$\ell(u)(t) = p(t)u(\tau(t)) - g(t)u(\mu(t))$$

Corollary

Let

$$p(t) \geq g(t) \qquad \text{for a. e. } t \in [a, b],$$
$$g(t)\big(\tau(t) - \mu(t)\big) \leq 0 \qquad \text{for a. e. } t \in [a, b],$$
$$p(t)\big(\tau(t) - t\big) \geq 0 \qquad \text{for a. e. } t \in [a, b].$$

Let, moreover, either

$$\int_a^b p(t)dt < 1$$

or

$$\int_t^{\tau(t)} p(s)ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a, b].$$

Then the conclusion of Theorem 4 holds.

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Corollary

Let

$$p(t) \geq g(t) \qquad \text{for a. e. } t \in [a,b],$$
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or μ be a nondecreasing continuous function with $\mu(t) \geq t$,

$$\int_t^{\mu(\mu(t))} \left(p(s) - g(s) + g(s) \int_{\tau(s)}^{\mu(s)} g(\xi)d\xi \right) ds \leq \frac{1}{e} \qquad \text{for a. e. } t \in [a,b].$$

Then the conclusion of Theorem 4 holds.

Define an operator $\varphi : C\big([a,b];\mathbb{R}\big) \to C\big([a,b];\mathbb{R}\big)$ as follows:

$$\varphi(v)(t) = v(a+b-t) \qquad \text{for } \ t \in [a,b], \qquad v \in C\big([a,b];\mathbb{R}\big).$$

Put

$$\widetilde{\ell}(v)(t) = -\ell(\varphi(v))(a+b-t) \qquad \text{for a. e. } \ t \in [a,b], \qquad v \in C\big([a,b];\mathbb{R}\big).$$

Then it can be easily verified that

$$\begin{aligned} \widetilde{\ell} \in \mathcal{S}_{ab}(a) \qquad &\Longleftrightarrow \qquad \ell \in \mathcal{S}_{ab}(b) \\ \widetilde{\ell} \in \mathcal{S}'_{ab}(a) \qquad &\Longleftrightarrow \qquad \ell \in \mathcal{S}'_{ab}(b) \\ \widetilde{\ell} \in \mathcal{P}^{+}_{ab} \qquad &\Longleftrightarrow \qquad -\ell \in \mathcal{P}^{-}_{ab} \\ \widetilde{\ell} \in \mathcal{P}^{-}_{ab} \qquad &\Longleftrightarrow \qquad -\ell \in \mathcal{P}^{+}_{ab} \end{aligned}$$