Monotone Solutions of the Cauchy Problem for Singular in Phase Variables Nonlinear Ordinary Differential Equations

Ivan Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: `kig@rmi.ge`

Let $-\infty < a < b < \infty$, $r > 0$, $\mathbb{R}_+ = [0, +\infty[$,

$$D = \Big\{(x_1, \dots, x_n) \in \mathbb{R}^n : \ \ 0 < x_1 < r, \dots, 0 < x_n < r\Big\},$$

and $f : [a, b] \times D \to \mathbb{R}_+$ be an integrable in the first and continuous in the last n arguments function. We consider the differential equation

$$u^{(n)} = f(t, u, \dots, u^{(n-1)}) \tag{1}$$

with the initial conditions

$$u^{(i-1)}(a) = 0 \ \ (i = 1, \dots, n). \tag{2}$$

We denote by $\widetilde{C}^{n-1}([a, b_0])$ the space of $(n-1)$-times continuously differentiable functions $u : [a, b_0] \to \mathbb{R}$ with an absolutely continuous derivatives of $(n-1)$-th order. A function $u \in \widetilde{C}^{n-1}([a, b_0])$, where $b_0 \in]a, b]$, is called **a solution of the differential equation** (1) **in the interval** $[a, b_0]$ if

$$0 < u^{(i-1)}(t) < r \ \text{ for } \ a < t < b_0 \ \ (i = 1, \dots, n),$$

and almost everywhere on $]a, b_0[$ this function satisfies equation (1).

If in some interval $[a, b_0] \subset [a, b[$, equation (1) has a solution satisfying the initial conditions (2), then we call problem (1), (2) **locally solvable**.

When a function f has nonintegrable singularities with respect to the time variable, the problem (1) (2) have been investigated with sufficient thoroughness (see, e.g., [1] and the references therein). However, when equation (1) is singular in phase variables, i.e., when

$$\lim_{x_1 + \dots + x_n \to 0} f(t, x_1, \dots, x_n) = +\infty,$$

the question on solvability of problem (1), (2) still remains open.

The theorem on the solvability of the problem (1), (2) deals exactly with this case.

More precisely, we consider this case where the function f in the domain $]a, b[\times D$ satisfies the inequality

$$\sum_{i=1}^{\ell} \frac{g_i(t)}{h_i(x_{n_i})} \le f(t, x_1, \dots, x_n) \le g(t, x_{n_1}, \dots, x_{n_\ell}). \tag{3}$$

Here $\ell \in \{1, \dots, n\}$, $n_i \in \{1, \dots, n\}$, $n_i < n_j$ for $i < j$,

$$D_\ell = \Big\{(y_1, \dots, y_\ell) : \ \ 0 < y_1 < r, \dots, 0 < y_\ell < r\Big\},$$

and $g :]a, b[\times D_\ell \to \mathbb{R}_+$ is an integrable in the first argument on $[a, b]$ and continuous nonincreasing in each of the last ℓ arguments function. As for $g_i :]a, b[\to \mathbb{R}_+$ $(i = 1, \dots, \ell)$ and $h_i :]0, r] \to]0, +\infty[$ $(i = 1, \dots, \ell)$, they are, respectively, integrable and continuous nondecreasing functions such that

$$\int_a^t g_i(s)\, ds > 0 \ \text{ for } \ a < t \le b, \quad \lim_{x \to 0} h_i(x) = 0 \ \ (i = 1, \dots, \ell). \tag{4}$$

Let

$$H_i(x)=\int_a^x h_i(y)\,dy \;\text{ for }\; 0\le x\le r \;\; (i=1,\dots,\ell), \tag{5}$$

$$\delta_i(t)=H_{i-1}^{-1}\bigg(\frac{1}{(n-n_i)!}\int_a^t (t-s)^{n-n_i}g_i(s)\,ds\bigg) \;\text{ for }\; a\le t\le b_0 \;\; (i=1,\dots,\ell), \tag{6}$$

where H_i^{-1} is the inverse function to H_i, and $b_0\in\,]a,b]$ is the number chosen so that

$$\int_a^{b_0}(b-s)^{n-n_i}g_i(s)\,ds<(n-n_i)!\,H_i(r) \;\; (i=1,\dots,\ell). \tag{7}$$

Theorem 1. *If along with* (3) *and* (4) *the condition*

$$\int_a^{b_0} g\big(t,\delta_1(t),\dots,\delta_\ell(t)\big)\,dt<+\infty \tag{8}$$

holds, then problem (1), (2) *is locally solvable.*

The particular cases of (1) are the differential equations

$$u^{(n)}=\sum_{i=1}^{\ell}\frac{g_i(t)}{h_i(u^{(n_i-1)})}\,f_i(t,u,\dots,u^{(n-1)}) \tag{9}$$

and

$$u^{(n)}=\sum_{i=1}^{\ell}\frac{(t-a)^{\mu_i}}{(u^{(n_i-1)})^{\nu_i}}\,f_i(t,u,\dots,u^{(n-1)}). \tag{10}$$

For these equations, Theorem 1 gives rise the following corollaries.

Corollary 1. *Let $f_i:\,]a,b[\,\times D\to[1,+\infty[$ $(i=1,\dots,\ell)$ be continuous, bounded functions, $g_i:[a,b]\to\mathbb{R}_+$ $(i=1,\dots,\ell)$ be integrable and $h_i:\,]0,r]\to[0,+\infty[$ $(i=1,\dots,\ell)$ be continuous nondecreasing functions, satisfying conditions* (4). *Let, moreover,*

$$\int_a^{b_0}\frac{g_i(t)}{h_i(\delta_i(t))}\,dt<+\infty \;\; (i=1,\dots,\ell),$$

where δ_i $(i=1,\dots,\ell)$ are the functions given by equalities (6). *Then problem* (9), (2) *is locally solvable.*

Corollary 2. *Let $\mu_i\in\mathbb{R}$, $\nu_i>0$ $(i=1,\dots,\ell)$, $c>0$, and $f_i:\,]a,b[\,\times D\to[c,+\infty[$ $(i=1,\dots,\ell)$ be continuous, bounded functions. Then for problem* (10), (2) *to be locally solvable it is necessary and sufficient that*

$$\mu_i>(n-n_i)\nu_i-1 \;\; (i=1,\dots,\ell). \tag{11}$$

Acknowledgement

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

References

[1] I. T. Kiguradze, *Some Singular Boundary Value Problems for Ordinary Differential Equations* (in Russian), Tbilisi University Press, Tbilisi, 1975.