Nonlinear Nonlocal Boundary Value Problems for Singular in a Phase Variable Second Order Differential Equations

Ivan Kiguradze

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia
E-mail: kig@rmi.ge

In a finite interval $]a,b[\,$, we consider the nonlinear differential equation

$$u'' = f(t,u) \tag{1}$$

with the nonlinear nonlocal boundary conditions of one of the following three types:

$$u(a) = \int\limits_a^b h_1(u(s))\,d\ell_1(s), \quad u(b) = \int\limits_a^b h_2(u(s))\,d\ell_2(u); \tag{2}$$

$$u(a) = \int\limits_a^b h_1(u(s))\,d\ell_1(s), \quad u(b) = \int\limits_a^b h_2(u(s))\,d\ell_0(u); \tag{3}$$

$$u(a) = \int\limits_a^b h_1(u(s))\,d\ell_1(s), \quad u'(b) = \int\limits_a^b h_2(u(s))\,d\ell_2(u). \tag{4}$$

Here, $f :]a,b[\times]0,+\infty[\to \mathbb{R}_-$ is a measurable in the first and continuous in the second argument function, $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ $(i = 1,2)$ are continuous functions, $\mathbb{R}_- =]-\infty,0]$, $\mathbb{R}_+ = [0,+\infty[\,$, $a < b_0 < b$, and $\ell_i : [a,b] \to \mathbb{R}_+$ $(i = 1,2)$ and $\ell_0 : [a,b_0] \to \mathbb{R}_+$ are nondecreasing functions such that

$$\ell_i(b) - \ell_i(a) = 1 \ \ (i = 1,2), \quad \ell_0(b_0) - \ell_0(a) = 1.$$

Let $C([a,b];\mathbb{R})$ be the space of continuous functions $u : [a,b] \to \mathbb{R}$ and let $\widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$ be the space of continuously differentiable functions $u :]a,b[\to \mathbb{R}$ whose first derivatives are absolutely continuous on $[a+\varepsilon, b-\varepsilon]$ for arbitrarily small $\varepsilon > 0$.

A function $u \in C([a,b];\mathbb{R}) \cap \widetilde{C}^1_{loc}(]a,b[\,;\mathbb{R})$ is said to be a **positive solution** of the equation (1) if

$$u(t) > 0 \ \text{ for } \ a < t < b$$

and

$$u''(t) = f(t,u(t)) \ \text{ for almost all } \ t \in]a,b[\,.$$

A positive solution u of the equation (1) is said to be a **positive solution of the problem** $(1),(k)$, where $k \in \{2,3\}$, (**of the problem** $(1),(4)$) if it satisfies the equalities (k) (has a finite limit $u'(b) = \lim\limits_{t\to b} u'(t)$ and satisfies the equalities (4)).

The theorems below on the existence of a positive solution of the problems $(1),(k)$ $(k = 2,3,4)$ deal with the cases where the function f in the domain $]a,b[\times]0,+\infty[$ satisfies the inequality

$$-p_1(t,x) - p_2(t,x)(1+x) \le f(t,x) \le -p_0(t,x), \tag{5}$$

where $p_i :]a,b[\times]0,+\infty[\to \mathbb{R}_+$ $(i=0,1,2)$ are measurable in the first and nonincreasing in the second argument functions, and the functions h_i $(i=1,2)$ satisfy one of the following three conditions:

$$\limsup_{x\to+\infty} \frac{h_i(x)}{x} \le r < 1 \quad (i=1,2), \tag{6}$$

$$\limsup_{x\to+\infty} \frac{h_1(x)}{x} \le r < 1, \quad h_2(x) \le x \ \text{ for } \ x\in\mathbb{R}_+, \tag{7}$$

$$\limsup_{x\to+\infty} \frac{h_1(x)}{x} + (b-a) \limsup_{x\to+\infty} \frac{h_2(x)}{x} \le r < 1. \tag{8}$$

We are mainly interested in the case

$$\lim_{x\to 0} p_i(t,x) = +\infty \ \text{ for } \ t\in I \quad (i=0,1,2),$$

where $I \subset [a,b]$ is a set of positive measure. In this case,

$$\lim_{x\to 0} f(t,x) = -\infty \ \text{ for } \ t\in I,$$

i.e. the equation (1) is singular in the phase variable.

The following theorems are valid.

Theorem 1. *If along with* (5) *and* (6) *the conditions*

$$0 < \int_a^b (t-a)(b-t)p_i(t,x)\,dt < +\infty \ \textit{ for } \ x>0 \quad (i=0,1), \tag{9}$$

$$\lim_{x\to+\infty} \int_a^b (t-a)(b-t)p_2(t,x)\,dt < (1-r)(b-a)$$

are fulfilled, then the problem (1), (2) *has at least one positive solution.*

Theorem 2. *If along with* (5), (7) *and* (9) *the condition*

$$\lim_{x\to+\infty} \int_a^b (t-a)(b-t)p_2(t,x)\,dt < (1-r)(b-b_0)$$

holds, then the problem (1), (3) *has at least one positive solution.*

Theorem 3. *If along with* (5) *and* (8) *the conditions*

$$0 < \int_a^b (t-a)p_i(t,x)\,dt < +\infty \ \textit{ for } \ x>0 \quad (i=0,1),$$

$$\lim_{x\to+\infty} \int_a^b (t-a)p_2(t,x)\,dt < (1-r)(b-a)$$

are fulfilled, then the problem (1), (4) *has at least one positive solution.*

Note that if the conditions of Theorem 1 or 2 (of Theorem 3) are fulfilled but

$$\int_a^b p_i(t,x)\,dt = +\infty \ \text{ for } \ x > 0 \ \ (i = 0, 1),$$

then the equation (1) has a nonintegrable singularity in the time variable at the point $t = a$ or $t = b$ (at the point $t = a$).

As an example, we consider the differential equation

$$u'' = -\sum_{k=1}^{n} \frac{f_k(t)}{q_k(u)}\, u^{\lambda_k} - \frac{f_0(t)}{q_0(u)}\,, \tag{10}$$

where

$$0 < \lambda_k \le 1 \ \ (k = 1, \dots, n),$$

$f_k :]a, b[\to \mathbb{R}_+$ $(k = 0, 1, \dots, n)$ are measurable functions, and $q_k :]0, +\infty[\to]0, +\infty[$ $(k = 0, 1, \dots, n)$ are continuous, nondecreasing functions such that

$$\lim_{x \to 0} q_k(x) = 0 \ \ (k = 0, 1, \dots, n).$$

Theorems 1–3 result in the following corollaries.

Corollary 1. *If*

$$0 < \int_a^b (t-a)(b-t) f_0(t)\,dt < +\infty, \quad \int_a^b (t-a)(b-t) f_k(t)\,dt < +\infty \ \ (k = 1, \dots, n), \tag{11}$$

$$\limsup_{x \to +\infty} \frac{h_i(x)}{x} < 1 \ \ (i = 1, 2),$$

and

$$\lim_{x \to +\infty} q_k(x) = +\infty \ \ (k = 1, \dots, n), \tag{12}$$

then the problem (10), (2) *has at least one positive solution.*

Corollary 2. *If along with* (11) *and* (12) *the condition*

$$\lim_{x \to +\infty} \frac{h_1(x)}{x} < 1, \ \ h_2(x) \le x \ \textit{ for } \ x > 0$$

holds, then the problem (10), (3) *has at least one positive solution.*

Corollary 3. *If*

$$0 < \int_a^b (t-a) f_0(t)\,dt < +\infty, \quad \int_a^b (t-a) f_k(t)\,dt < +\infty \ \ (k = 1, \dots, n),$$

$$\limsup_{x \to +\infty} \frac{h_1(x)}{x} + (b-a) \limsup_{x \to +\infty} \frac{h_2(x)}{x} < 1,$$

and the condition (12) *is fulfilled, then the problem* (10), (4) *has at least one positive solution.*

Acknowledgements

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).