The Nonlinear Kneser Problem for Singular in Phase Variables Second Order Nonlinear Differential Equations

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Let
$$D = \{(t,x,y):\ t>0,\ x>0,\ y<0\}, \quad R_+ = [0,+\infty[,$$
and let $f : D \to R_+$ be a continuous function. A continuous function $u : R_+ \to R_+$ is said to be **the Kneser solution of the equation**
$$u'' = f(t,u,u') \tag{1}$$
if it is twice continuously differentiable in the interval $]0,+\infty[$ and in this interval satisfies the inequalities
$$u(t) > 0, \quad u'(t) < 0$$
and the differential equation (1).

We consider the problem on the existence of a Kneser solution of equation (1), satisfying the condition
$$\varphi(u) = c, \tag{2}$$
where $\varphi : C([0,a];R_+) \to R_+$ is a continuous, nondecreasing functional, $a > 0$, and $c > 0$.

We name this problem the nonlinear Kneser problem since it was first studied by Kneser in the case, where $\varphi(u) \equiv u(0)$, $f(t,x,y) \equiv f_0(t,x)$, and $f_0 : R_+ \times R_+ \to R_+$ is a continuous function such that $f_0(t,0) \equiv 0$.

The Kneser type problems for nonlinear differential equations and systems, not having singularities in phase variables, are studied in detail (see, [1]–[8], and the references therein).

We are interested in the case, where the function f satisfies the inequality
$$g_0(t) \le x^\lambda |y|^\mu f(t,x,y) \le g_1(t)$$
in the domain D. Here λ and μ are nonnegative constants, $\lambda + \mu > 0$, and $g_i :]0,+\infty[\to]0,+\infty[$ $(i = 0,1)$ are continuous functions. In this case
$$\lim_{x\to 0,\, y\to 0} f(t,x,y) = +\infty \quad \text{for} \ \ t > 0,$$
i.e. equation (1) has singularities in phase variables.

The Kneser problem for the differential equation with a singularity in one of the phase variables first was investigated by I. Kiguradze [9]. However, in this paper there is considered not the general differential equation but the Emden–Fowler type higher order differential equation $u^{(n)} = p(t)u^{-\lambda}$.

A Kneser solution u of equation (1) is called **vanishing at infinity** if $\lim\limits_{t\to+\infty} u(t) = 0$, and it is called **remote from zero** if $\lim\limits_{t\to+\infty} u(t) > 0$.

Theorem 1 *If equation* (1) *has a Kneser solution, then*

$$\int\limits_t^{+\infty} g_0(s)ds < +\infty \;\; for \;\; t > 0, \quad \int\limits_0^{+\infty} \left(\int\limits_t^{+\infty} g_0(s)ds \right)^{\frac{1}{\mu+1}} dt < +\infty, \tag{3}$$

and $u(t) > v_0(t;\delta)$ *for* $t \geq 0$, *where* $\delta = \lim\limits_{t\to+\infty} u(t)$,

$$v_0(t;\delta) = \left[\delta^\nu + (1+\mu)^{\frac{1}{1+\mu}} \nu \int\limits_t^{+\infty} \left(\int\limits_s^{+\infty} g_0(x)dx \right)^{\frac{1}{1+\mu}} ds \right]^{\frac{1}{\nu}}, \quad and \;\; \nu = \frac{1+\lambda+\mu}{1+\mu}.$$

Corollary 1 *If condition* (3) *is fulfilled and* $c \leq \varphi(v_0(\cdot;0))$, *then equation* (1) *has no Kneser solution, satisfying condition* (2).

Theorem 2 *If*

$$\int\limits_t^{+\infty} g_1(s)ds < +\infty \quad for \;\; t > 0, \quad \int\limits_0^{+\infty} \left(\int\limits_t^{+\infty} g_1(s)ds \right)^{\frac{1}{1+\mu}} dt < +\infty, \tag{4}$$

then for any positive number δ *equation* (1) *has at least one Kneser solution* u *such that* $u(t) \to \delta$ *as* $t \to +\infty$.

According to Corollary 1, for small c problem (1), (2) has no Kneser solution. Thus we can expect the solvability of that problem only for large c.

Suppose condition (4) holds. Then obviously condition (3) is satisfied as well. We introduce the function

$$v_1(t;\delta) = \delta + \int\limits_t^{+\infty} \left[(1+\mu) \int\limits_s^{+\infty} \frac{g_1(x)}{v_0^\lambda(x;\delta)} dx \right]^{\frac{1}{1+\mu}} ds \;\; \text{for} \;\; t \geq 0, \;\; \delta > 0,$$

and the number $c_0 = \inf\{\varphi(v_1(\cdot;\delta) : \; \delta > 0\}$.

Theorem 3 *Let the function* g_1 *satisfy condition* (4), *and*

$$\lim_{x\to+\infty} \varphi(x) = +\infty. \tag{5}$$

If, moreover, $c > c_0$, *then problem* (1),(2) *has at least one Kneser solution.*

Theorem 4 *Let $g_1(t) \equiv \ell g_0(t)$, $\ell = \text{const} \geq 1$, and let there exist numbers α and β such that*

$$\liminf_{t\to 0} \left(t^\alpha g_0(t)\right) > 0, \quad \limsup_{t\to 0} \left(t^\alpha g_0(t)\right) < +\infty,$$
$$\liminf_{t\to +\infty} \left(t^\beta g_0(t)\right) > 0, \quad \limsup_{t\to +\infty} \left(t^\beta g_0(t)\right) < +\infty.$$

Let, moreover, the functional φ satisfy condition (5). *Then the following assertions are equivalent:*

(*i*) $\alpha < 2 + \mu$, $\beta > 2 + \mu$;

(*ii*) *equation* (1) *has at least one remote from zero Kneser solution;*

(*iii*) *equation* (1) *has at least one vanishing at infinity Kneser solution;*

(*iv*) *for any sufficiently large $c > 0$, problem* (1), (2) *has at least one Kneser solution.*

Acknowledgements

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

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