On special solutions to Emden–Fowler type differential equations [1].

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1 Introduction

For the higher-order Emden–Fowler type differential equation

$$y^{(n)} + p_0\,|y|^k \operatorname{sgn} y = 0, \quad n > 2, \quad k \in \mathbb{R}, \quad k > 1, \quad p_0 \neq 0, \tag{1}$$

the existence of oscillatory and non-oscillatory solutions with special asymptotic behavior is proved. This yields the existence of solutions with arbitrary number of zeros.

A lot of results on the asymptotic behavior of solutions to (1) are described in detail in [1]. Results on the existence of solutions with special asymptotic behavior are contained in [2]–[9].

Hereafter we use the notation

$$\alpha = \frac{n}{k-1}. \tag{2}$$

2 Existence of positive solutions with special asymptotic behavior

For equation (1) with $p_0 = -1$ it was proved [4] that for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (1) has a solution of the form

$$y = (x^* - x)^{-\alpha}\; h(\log(x^* - x)),$$

where α is defined by (2) and h is a positive periodic non-constant function on $\mathbf{R}$.

A similar result was also proved [4] about Kneser solutions, i. e. those satisfying $y(x) \to 0$ as $x \to \infty$ and $(-1)^j\, y^{(j)}(x) > 0$ for $0 \le j < n$. Namely, if $p_0 = (-1)^{n-1}$, then for any N and $K > 1$ there exist an integer $n > N$ and $k \in \mathbf{R}$ such that $1 < k < K$ and equation (1) has a solution of the form

$$y(x) = (x - x_*)^{-\alpha}\; h(\log(x - x_*)),$$

where h is a positive periodic non-constant function on $\mathbf{R}$.

Still it was not clear how large n should be for the existence of that type of positive solutions.

Theorem 1 ([8]) *If $12 \le n \le 14$, then there exists $k > 1$ such that equation (1) with $p_0 = -1$ has a solution $y(x)$ such that*

$$y^{(j)}(x) = (x^* - x)^{-\alpha-j}\, h_j(\log(x^* - x)), \qquad j = 0, 1, \ldots, n-1,$$

where α is defined by (2) and h_j are periodic positive non-constant functions on $\mathbf{R}$.

Remark 1 *Computer calculations give approximate values of α providing the existence of the above-type solutions. They are, with the corresponding values of k, as follows:*
if $n = 12$, then $\alpha \approx 0.56$, $k \approx 22.4$;
if $n = 13$, then $\alpha \approx 1.44$, $k \approx 10.0$;
if $n = 14$, then $\alpha \approx 2.37$, $k \approx 6.9$.

Corollary 1.1 ([8]) *If $12 \le n \le 14$, then there exists $k > 1$ such that equation (1) with $(-1)^n p_0 < 0$ has a Kneser solution $y(x)$ satisfying*

$$y^{(j)}(x) = (x - x_0)^{-\alpha-j}\, h_j(\log(x - x_0)), \quad j = 0, 1, \ldots, n-1,$$

with periodic positive non-constant functions h_j on $\mathbf{R}$.

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3 Existence of special oscillatory solutions

Theorem 2 *For any integer $n > 2$ and real $k > 1$ there exists a non-constant oscillatory periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}}(x^* - x)^{-\alpha} h\left(\log(x^* - x)\right), \quad -\infty < x < x^*, \tag{3}$$

is a solution to equation (1).

Corollary 2.1 *For any integer even $n > 2$ and real $k > 1$ there exists a non-constant oscillatory periodic function $h(s)$ such that for any $p_0 > 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = p_0^{\frac{1}{k-1}}(x - x^*)^{-\alpha} h\left(\log(x - x^*)\right), \quad x^* < x < \infty, \tag{4}$$

is a solution to equation (1).

Corollary 2.2 *For any integer odd $n > 2$ and real $k > 1$ there exists a non-constant oscillatory periodic function $h(s)$ such that for any $p_0 < 0$ and $x^* \in \mathbb{R}$ the function*

$$y(x) = |p_0|^{\frac{1}{k-1}}(x - x^*)^{-\alpha} h\left(\log(x - x^*)\right), \quad x^* < x < \infty, \tag{5}$$

is a solution to equation (1).

4 Existence of oscillatory solutions with prescribed number of zeros

(with V.Rogachev)

Theorem 3 *For any integer $m \geq 2$ and even $n > 2$, and any real $k > 1$, $p_0 > 0$, $-\infty < a < b < +\infty$, equation* (1) *has a solution defined on the segment $[a, b]$, vanishing at its end points a and b, and having exactly m zeros on the segment $[a, b]$.*

Theorem 4 *For any integer $m \geq 2$ and odd $n > 2$, and any real $k > 1$, $p_0 \neq 0$, $-\infty < a < b < +\infty$, equation* (1) *has a solution defined on the segment $[a, b]$, vanishing at its end points a and b, and having exactly m zeros on the segment $[a, b]$.*

Theorem 5 *For any integer $n > 2$ and real $k > 1$, $p_0 > 0$, $-\infty < a < b < +\infty$, equation* (1) *has a solution defined on the half-open interval $[a, b)$, vanishing at its end point a and having a countable number of zeros on the interval $[a, b)$.*

Theorem 6 *For any integer odd $n > 2$ and real $k > 1$, $p_0 < 0$, $-\infty < a < b < +\infty$, equation* (1) *has a solution defined on the half-open interval $(a, b]$, vanishing at its end point b and having a countable number of zeros on the interval $(a, b]$.*

Remark 2 *The same results for $n = 3, 4$ were published in [9].*

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