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On the well-posedness of two-point weighted boundary value problems for second order nonlinear differential equations with strong singularities

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Let $-\infty < a < b < +\infty$ and $f\colon]a,b[\times\mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the second order nonlinear differential equation

$$u'' = f(t,u) \tag{1}$$

with weighted boundary conditions of one of the following two types:

$$\limsup_{t\to a} \frac{|u(t)|}{(t-a)^{\alpha}} < +\infty, \quad \limsup_{t\to b} \frac{|u(t)|}{(b-t)^{\beta}} < +\infty \tag{2}$$

and

$$\limsup_{t\to a} \frac{|u(t)|}{(t-a)^{\alpha}} < +\infty, \quad \lim_{t\to b} u'(t) = 0, \tag{3}$$

where $\alpha \in]0,1[$ and $\beta \in]0,1[$.

Below we give the results from [4] which contain unimprovable in a certain sense conditions guaranteeing the well-posedness of the problems (1), (2) and (1), (3). These results cover the case, where Eq. (1) at the points a and b has strong singularities in the sense of R. P. Agarwal and I. Kiguradze [1, 2, 3], i.e. the case, where for any $t_0 \in]a,b[$ and $x > 0$ the condition $\int_a^{t_0}(t-a)\big[|f(t,x)| - f(t,x)\operatorname{sgn} x\big]\,dt = +\infty$ or the condition $\int_{t_0}^{b}(b-t)\big[|f(t,x)| - f(t,x)\operatorname{sgn} x\big]\,dt = +\infty$ is satisfied.

Let

$$G_0(t,s) = \begin{cases} \dfrac{(s-a)(t-b)}{b-a} & \text{for } a \le s \le t \le b, \\ \dfrac{(t-a)(s-b)}{b-a} & \text{for } a \le t < s \le b, \end{cases} \qquad G_1(t,s) = \begin{cases} a-s & \text{for } a \le s \le t \le b, \\ a-t & \text{for } a \le t < s \le b. \end{cases}$$

For any continuous function $h\colon\ ]a,b[\to \mathbb{R}$, we assume

$$\nu_{\alpha,\beta}(h) = \sup\left\{ (t-a)^{-\alpha}(b-t)^{-\beta} \int_a^b |G_0(t,s)h(s)|\,ds : a < t < b \right\},$$

$$\nu_{\alpha}(h) = \sup\left\{ (t-a)^{-\alpha} \int_a^b |G_1(t,s)h(s)|\,ds : a < t < b \right\}.$$

Definition 1. The problem (1), (2) (the problem (1), (3)) is said to be **well-posed** if for any continuous function $h\colon\ ]a,b[\to\mathbb{R}$, satisfying the condition $\nu_{\alpha,\beta}(h)<+\infty$ $\big(\nu_\alpha(h)<+\infty\big)$, the perturbed differential equation

$$v''=f(t,v)+h(t) \tag{4}$$

has a unique solution, satisfying the boundary conditions (2) (the boundary conditions (3)), and there exists a positive constant r, independent of the function h, such that in the interval $]a,b[$ the inequality

$$|u(t)-v(t)|\leq r\nu_{\alpha,\beta}(h)(t-a)^\alpha(b-t)^\beta \quad \Big(|u(t)-v(t)|\leq r\nu_\alpha(h)(t-a)^\alpha\Big)$$

is satisfied, where u and v are the solutions of (1), (2) and (4), (2) (of (1), (3) and (4), (3)), respectively.

Theorem 1. *Let there exist a continuous function* $p\colon\ ]a,b[\to[0,+\infty[$ *such that*

$$f(t,x)-f(t,y)\geq-(t-a)^{-\alpha}(b-t)^{-\beta}p(t)(x-y) \quad \text{for } a<t<b,\ x>y.$$

If, moreover, $\nu_{\alpha,\beta}(p)<1$, $\nu_{\alpha,\beta}(f(\cdot,0))<+\infty$, *then the problem* (1), (2) *is well-posed.*

Theorem 2. *Let there exist a continuous function* $p\colon\ ]a,b[\to[0,+\infty[$ *such that*

$$f(t,x)-f(t,y)\geq-(t-a)^{-\alpha}p(t)(x-y) \quad \text{for } a<t<b,\ x>y.$$

If, moreover, $\nu_\alpha(p)<1$, $\nu_\alpha(f(\cdot,0))<+\infty$, *and* $\int_t^b f^*(s,x)ds<+\infty$ *for* $a<t<b$, $x>0$, *where* $f^*(t,x)=\max\{|f(t,y)|:|y|\leq x\}$, *then the problem* (1), (3) *is well-posed.*

The condition $\nu_{\alpha,\beta}(p)<1$ (the condition $\nu_\alpha(p)<1$) in Theorem 1 (in Theorem 2) is unimprovable and it cannot be replaced by the condition $\nu_{\alpha,\beta}(p)\leq1$ (by the condition $\nu_\alpha(p)\leq1$).

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