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On Izobov's problem for a nonlinear third-order differential equation

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Abstract

Sufficient and necessary conditions for existence of Kneser solutions vanishing at infinity of a nonlinear third-order differential equation with singular nonlinearity will be discussed.

Introduction

Consider the differential equation

$$y^{(n)} = p(x)|y|^{\lambda-1}\,y, \quad 0<\lambda<1,\;\; x\ge 0 \tag{1}$$

with

$$(-1)^{(n)}p(x)\ge 0, \quad x\ge 0. \tag{2}$$

Definition 1 ([1], [4]). A solution $y(x)$ of (1) is called a *Kneser solution vanishing at infinity* if

$$(-1)^{(i)}y^{(i)}(x)>0, \tag{3}$$

$$|y^{(i-1)}(x)|\ \downarrow\ 0, \quad x\to\infty, \quad i=1,2,\dots,n. \tag{4}$$

In [2] a sufficient condition was obtained for existence of solutions $y(x)$ satisfying (3), (4):

Theorem 1. *If a continuous function $p(x)$ satisfies the condition*

$$\int_0^{+\infty}\tau^{n-1}|p(\tau)|d\tau<\infty \tag{5}$$

then (1) *has solutions $y(x)$ such that* (3), (4) *hold.*

Later in [3] N. Izobov proved that (5) is not necessary if $n=2$:

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Theorem 2 (N. Izobov)**.** *Let $n = 2$. For any $\mu \geq \frac{1}{(n-1)\lambda+1}$ and any function $\varphi(x) > 0$ there exists a piecewise continuous non-negative function $p(x)$ satisfying the condition*

$$\int_0^{+\infty} p^{\mu}(\tau)\varphi(\tau)d\tau = \infty \tag{6}$$

such that equation (1) *has a Kneser solution $y(x)$ vanishing at infinity.*

Corollary 1 (N. Izobov)**.** *Let $n = 2$. There exists a piecewise continuous non-negative function $p(x)$ satisfying the condition*

$$\int_0^{+\infty} \tau^{n-1}|p(\tau)|d\tau = \infty \tag{7}$$

such that equation (1) *has a solution $y(x)$ with* (3), (4).

Problem (N. Izobov)**:** Is it possible to prove for $n > 2$ the analogue of Theorem 2?

A partial answer is given here for $n = 3$.

Main result

Theorem 3. *Suppose $n = 3$ and $0 < \lambda < 1$. Than for any $\mu > \frac{1}{2\lambda+1}$ and any continuous positive function $\varphi(x)$ $x \geq 0$, there exists a smooth negative function $p(x)$ such that the condition*

$$\int_0^{+\infty} |p(\tau)|^{\mu}\varphi(\tau)d\tau = \infty \tag{8}$$

holds and equation (1) *has a solution satisfying conditions* (3), (4).

In fact the following result is proved:

Theorem 4. *Suppose $n = 3$,$0 < \lambda < 1$, $\varphi(x)$ is a continuous positive function for $x \geq 0$, $\mu > \frac{1}{2\lambda+1}$. Then there exists a C^{∞} function $y(x)$, $x \geq 0$, such that*

$$|y^{(i)}(x)| \downdownarrows 0, \quad x \to \infty,\ i = 0, 1, 2, \tag{9}$$

and

$$\int_0^{+\infty} |y'''(\xi)|^{\mu}|y(\xi)|^{-\mu\lambda}\varphi(\xi)d\xi = \infty. \tag{10}$$

Remark 1. To generalize Theorem 2 the inequality $\mu \geq \frac{1}{2\lambda+1}$ is needed. We prove Theorem 3 for $\mu > \frac{1}{2\lambda+1}$ only, so Izobov's problem is partially solved. However we prove existence of a smooth negative function $p(x)$, which is a piecewise continuous non-positive function in Theorem 2.

Remark 2. Asymptotic behavior of solutions of the third-order equation (1) is described in [5, 6].

References

[1] I. T. Kiguradze, *On oscillatory and monotone solutions of ordinary differential equations*, Arch. Math. **14** (1978), No. 1, 21–44.

[2] G. G. Kvinikadze, *Some remarks on solutions of the Kneser problem*, Differentsial'nye Uravneniya **14** (1978), No. 10, 1775–1783, (in Russian).

[3] N. A. Izobov, *On Kneser solutions*, Differentsial'nye Uravneniya **21** (1985), No. 4, 581–588, (in Russian).

[4] I. T. Kiguradze, T. A. Chanturia, *Asymptotic properties of solutions of nonautonomous ordinary differenyial equations*, Kluwer Academic Publishers, Dordreht-Boston-London, 1993.

[5] I. V. Astashova, *Application of dynamical systems to the study of asymptotic properties of solutions to nonlinear higher-order differential equations*, J. Math. Sci., New York **126** (2005), No. 5, 1361–1391.

[6] I. V. Astashova, *Qualitative properties of solutions to quasilinear ordinary differential equations*, M.: MESI, 2010, 242 p., (in Russian), (ISBN 978-5-7764-0647-8).